Second Order Necessary and Sufficient Conditions for Efficiency in Multiobjective Programming Problems

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Abstract

This paper is concerned with multiobjective programming problem with inequality constraints. A generalized Abadie's constraint qualification for second-order tangent sets is used, and based on the later we give second-order necessary and sufficient conditions for efficiency.

Keywords: Multiobjective Programming, Efficient Solutions, Constraint Qualifications, Second-order Necessary and Sufficient Conditions.

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1 Introduction

In multiobjective programming problem, the first-order necessary and / or sufficient condition for efficiency have been studied extensively in the literature [5, 6, 8, 9]. But little work concerns second-order necessary and sufficient conditions for a feasible solution to be an efficient solution.

In this paper, we consider the multiobjective programming problems with inequality constraints. A generalized Abadie's constraint qualification for second-order tangent sets is used, and based on the later we shall give second-order necessary and sufficient conditions for efficiency.

This paper is organized as follow. In section 1, we shall formulate a multiobjective programming problem with inequality constraints, give some definitions and basic results, which are used throughout the paper. In section 3, we shall define the second-order tangent sets, and use the generalized Abadie's constraint qualification to derive second-order necessary conditions for a feasible solution to be efficient to the multiobjective programming problems. In section 4, we shall give sufficient conditions for efficiency.

2 Preliminaries

Consider the following multiobjective programming problem

$$(P)\quad \begin{array}{ll} \text{minimize} & (f_1(x), ..., f_l(x)) \\ \text{s.t} & g_j(x) \leq 0, j = 1, ..., m \end{array}$$

where f_i $(i \in L = \{1, ..., l\})$ and g_j $(j \in M = \{1, ..., m\})$ are twice differentiable on $I\!R^n$. Before describing the concept of an efficient solution, we describe our notations. For any vector y, we denote the Jacobian (resp. the Hessian) of f and g at $x \in R^n$ by $\nabla f(x)$ and $\nabla g(x)$ (resp. $\nabla^2 f(x)(y, y)$ and $\nabla^2 g(x)(y, y)$) and

$$A = \{x \in I\!R^n \mid g_j(x) \leq 0 j = 1, ..., m\}$$

We denote,

$$\begin{array}{lll} f(x) \leqq f(\bar{x}) & \text{implying} & f_i(x) \leqq f_i(\bar{x}), \quad i = 1, ..., l, \\ f(x) \leq f(\bar{x}) & \text{implying} & f_i(x) \leqq f_i(\bar{x}), \ and \ f(x) \neq f(\bar{x}), \\ f(x) < f(\bar{x}) & \text{implying} & f_i(x) < f_i(\bar{x}), \quad i = 1, ..., l. \end{array}$$

and for $l = 2$,

$$f(x) \leqq_{lex} f(\bar{x}) \quad \text{implying} \quad \left\{ \begin{array}{l} f_1(x) < f_1(\bar{x}) \\ \text{or } f_1(x) = f_1(\bar{x}) \text{ and } f_2(x) \leqq f_2(\bar{x}) \end{array} \right.$$

$$f(x) <_{lex} f(\bar{x}) \quad \text{implying} \quad \left\{ \begin{array}{l} f_1(x) < f_1(\bar{x}) \\ \text{or } f_1(x) = f_1(\bar{x}) \text{ and } f_2(x) < f_2(\bar{x}) \end{array} \right.$$

the subscription lex is an abbreviation for lexicographic order.

Definition 2.1: A point $\bar{x} \in A$ is called an efficient solution to Problem (P), if there is no $x \in A$ such that $f(x) \leq f(\bar{x})$.

Let $\bar{x} \in A$ be any feasible solution to Problem (P), and let E be the subset of indices defined by

$$E \equiv \{ j \in \{1, 2, ..., m\} \mid g_j(\bar{x}) = 0 \} \tag{1}$$

Definition 2.2: The tangent cone to A at $\bar{x} \in A$ is the set defined by

$$T_1(A, \bar{x}) \equiv \{ y \in I\!R^n \mid \exists x^n \in A, \exists t_n \longrightarrow 0^+ \text{ such that } x^n = \bar{x} + t_n y + o(t_n) \} \tag{2}$$

Where $o(t_n)$ is a vector satisfying $\dfrac{\|o(t_n)\|}{t_n} \longrightarrow 0^+$.

Definition 2.3: The linearizing cone to A at $\bar{x} \in A$ is the set defined by

$$K_1 \equiv \{ y \in I\!R^n \mid \nabla g_j(\bar{x}) y \leqq 0, \ j \in E \} \tag{3}$$

3 Second-Order Necessary Conditions

Following Kawasaki [4], we define two kinds of second-order approximation sets to the feasible region. They can be considered as extensions of $T_1(A,\bar{x})$ and K_1 respectively.

Definition 3.1: The second-order tangent set to A at $\bar{x} \in A$ is the set defined by

$$T_2(A,\bar{x}) \equiv \{(y,z) \in I\!R^{2n} \mid \exists x^n \in A,\ \exists t_n \longrightarrow 0^+ \text{ such that } x^n = \bar{x} + t_n y + \frac{1}{2}t_n^2 z + o(t_n^2)\,\}$$

Where $o(t_n^2)$ is a vector satisfying $\dfrac{\|o(t_n^2)\|}{t_n^2} \longrightarrow 0^+$.

Definition 3.2: The second-order linearizing set to A at $\bar{x}$ is the set defined by

$$L_2 \equiv \{(y,z) \in I\!R^{2n} \mid (\nabla g_j(\bar{x})y, \nabla g_j(\bar{x})z + \nabla^2 g_j(\bar{x})(y,y))^T \leqq_{lex} (0,0)^T,\ j \in E,\}$$

The y-sections of L_2 and $T_2(A,\bar{x})$ will be denoted by $L_2(y)$ and $T_2(A,\bar{x})(y)$, respectively. That is,

$$L_2(y) = \{z \in I\!R^n \mid (y,z) \in L_2\} \qquad T_2(A,\bar{x})(y) = \{z \in I\!R^n \mid (y,z) \in T_2(A,\bar{x})\}$$

Lemma 3.1: [4] Let $\bar{x}$ be any feasible solution to problem (P). Then we have,

$$T_2(A,\bar{x}) \subset L_2$$

Second-order constraint qualification: A is said to satisfy the second-order Abadie's constraint qualification at $\bar{x} \in A$ if

$$L_2 \subset T_2(A,\bar{x}) \tag{4}$$

we denote simply (4) by second-order (ACQ).

Incidently, a first-order sufficient conditions for efficiency is that the following system has no zero solution y

$$\begin{array}{l} \nabla f(\bar{x})y \leqq 0, \\ \nabla g_E(\bar{x})y \leqq 0. \end{array} \tag{5}$$

and the condition of Kuhn-Tucker type for efficiency is equivalent [8] to the inconsistency of the following system:

$$\begin{array}{l} \nabla f(\bar{x})y < 0, \\ \nabla g_E(\bar{x})y \leqq 0. \end{array} \tag{6}$$

The gap between (5) and (6) is caused by the following directions:

$$\begin{array}{l} \nabla f(\bar{x})y \leqq 0 \\ \nabla f_i(\bar{x})y = 0, \text{ at least one } i \\ \nabla g_E(\bar{x})y \leqq 0. \end{array} \tag{7}$$

A direction y which satisfies (7) is called a critical direction.

For the sake of simplicity, we will use the following notations:

$$F_i(y,z) = (\nabla f_i(\bar{x})y, \nabla f_i(\bar{x})z + \nabla^2 f_i(\bar{x})(y,y))^T,$$
$$G_j(y,z) = (\nabla g_j(\bar{x})y, \nabla g_j(\bar{x})z + \nabla^2 g_j(\bar{x})(y,y))^T.$$

As an essentiel tool for the the proof of the second-order necessary conditions for efficiency we need the following lemma.

Lemma 3.2: Let $\bar{x} \in A$ be an efficient solution to problem (P). Then there is no $(y,z) \in T_2(A,\bar{x})$ with $F(y,z) <_{lex} 0$.

Where $F(y,z) <_{lex} 0$ implying $F_i(y,z) <_{lex} (0,0)^T$, $\forall i$.

Proof. Let $\bar{x}$ be an efficient solution to problem (P). We fix an arbitrary $(y,z) \in T_2(A,\bar{x})$ and, we assume that $F_i(y,z) <_{lex} (0,0)^T$, $\forall i$. Then, there exist $x^n \in A$ and $t_n \longrightarrow 0^+$ such that

$$x^n = \bar{x} + t_n y + \frac{1}{2}t_n^2 z + o(t_n^2).$$

By Taylor's expansion, for each i we have

$$f_i(x^n) = f_i(\bar{x}) + t_n \nabla f_i(\bar{x})y + \frac{1}{2}t_n^2(\nabla f_i(\bar{x})z + \nabla^2 f_i(\bar{x})(y,y)) + o(t_n^2) \tag{8}$$

• if $\nabla f_i(\bar{x})y < 0$, from (8) we have:

$$f_i(x^n) = f_i(\bar{x}) + t_n(\nabla f_i(\bar{x})y + \theta_i^n) \quad \text{with} \quad \lim_{n\to\infty} \theta_i^n = 0$$

Hence, there exists N_i such that $|\theta_i^n| < -\nabla f_i(\bar{x})y$ for $n \geqq N_i$.

• if $\nabla f(\bar{x})y = 0$, hence $\nabla f(\bar{x})z + \nabla^2 f(\bar{x})(y,y) < 0$ and from (8) we have:

$$f_i(x^n) = f_i(\bar{x}) + \frac{1}{2}t_n^2(\nabla f_i(\bar{x})z + \nabla^2 f_i(\bar{x})(y,y) + \delta_i^n) \quad \text{with} \quad \lim_{n\to\infty} \delta_i^n = 0$$

Hence, there exists M_i such that $|\delta_i^n| < -(\nabla f_i(\bar{x})z + \nabla^2 f_i(\bar{x})(y,y))$ for $n \geqq M_i$.

Finaly,

if $\nabla f_i(\bar{x})y < 0$ we take $K_i = N_i,\ q_i = \nabla f_i(\bar{x})y$ and $\gamma_i^n = \theta_i^n$.

if $\nabla f_i(\bar{x})y = 0$ we take $K_i = M_i$, $q_i = \nabla f_i(\bar{x})z + \nabla^2 f_i(\bar{x})(y,y))$ and $\gamma_i^n = \delta_i^n$.

Hence,

$$f_i(x^n) = f_i(\bar{x}) + r_n(q_i + \gamma_i^n) \quad \text{with} \quad \lim_{n\to\infty} \gamma_i^n = 0.$$

Where $r_n = t_n$ if $\nabla f_i(\bar{x})y < 0$ and $r_n = \frac{1}{2}t_n^2$ if $\nabla f_i(\bar{x})y = 0$.

Let $K = \max_{1 \leqq i \leq l} K_i$, then $f(x^n) < f(\bar{x})$ for $n \geqq K$. Which is a contradiction. □

Now, we are in a position to state the primal form of our second-order necessary conditions.

Theorem 3.1: Let $\bar{x}$ be an efficient solution to problem (P). Assume that the second-order (ACQ) holds at $\bar{x} \in A$. Then, the following system has no solution (y, z):

$$\begin{array}{ll} F_i(y,z) <_{lex} 0, & \forall i \\ G_j(y,z) \leqq_{lex} 0, & \forall j \in E. \end{array} \tag{9}$$

Proof. Let (y, z) be any element of $T_2(A, \bar{x})$, then, there exist $x^n \in A$ and $t_n \longrightarrow 0^+$ such that

$$x^n = \bar{x} + t_n y + \frac{1}{2}t_n^2 z + o(t_n^2)$$

by Taylor's expansion,

$$f(x^n) = f(\bar{x}) + t_n \nabla f(\bar{x})y + \frac{1}{2}t_n^2(\nabla f(\bar{x})z + \nabla^2 f(\bar{x})(y,y)) + o(t_n^2)$$

Which implies, $(\nabla f(\bar{x})y, \nabla f(\bar{x})z + \nabla^2 f(\bar{x})(y,y)) \in T_2(f(A), f(\bar{x}))$.

Since $\bar{x}$ is an efficient solution to Problem (P) and by lemma 3.2,

$$F(y,z) \not<_{lex} 0,$$

where $F(y,z) <_{lex} 0$ implying $F_i(y,z) <_{lex} 0, \quad \forall i$.

By assymption, we have

$$T_2(A, \bar{x}) = L_2$$

Hence, the following system has no solution (y, z):

$$\begin{array}{l} F_i(y,z) <_{lex} 0, \forall i, \\ G_j(y,z) \leqq_{lex} 0, \forall j \in E. \end{array}$$

□

In the following, for simplicity, we will denote (9) by

$$F(y,z) <_{lex} 0, \quad G_E(y,z) \leqq_{lex} 0.$$

It may be noted that theorem 3.2 contains the first-order optimality conditions for efficiency [6, 8, 9]. In fact, by taking $y = 0$, they are embedded in (9).

Consider the following multiobjective programming problem :

$$\begin{array}{ll} \text{minimize} & (f_1(x_1,x_2), f_2(x_1,x_2)) = (x_1,x_2) \\ \text{s.t} & g_1(x_1,x_2) = -x_1^2 - x_2 \leq 0 \end{array}$$

Then $(\bar{x}_1, \bar{x}_2)^T = (0,0)^T$ satisfy the first order necessary conditions: the following system is inconsistent

$$\begin{array}{l} \nabla f_1(\bar{x})y < 0, \\ \nabla f_2(\bar{x})y < 0, \\ \nabla g_1(\bar{x})y \leq 0. \end{array}$$

Which is

$$\begin{array}{l} \nabla f_1(\bar{x})y = y_1 < 0, \\ \nabla f_2(\bar{x})y = y_2 < 0, \\ \nabla g_1(\bar{x})y = -y_2 \leq 0, \end{array}$$

and we can not say any things about the efficiency of $\bar{x}$. But if we use our second-order necessary conditions: the system

$$\begin{array}{l} F_1(y,z) = (y_1,z_1) <_{lex} (0,0) \\ F_2(y,z) = (y_2,z_2) <_{lex} (0,0) \\ G_1(y,z) = (-y_2, -z_2 - 2y_1^2) \leqq_{lex} (0,0) \end{array}$$

have $(y,z) = ((-1,0),(0,-1))$ as solution. Hence by theorem 3.1, $\bar{x}$ is not efficient.

Now, we shall state the dual form of theorem 3.1.

Theorem 3.2: Let $\bar{x}$ satisfy the assumptions of theorem 3.1. Then, for each critical direction y, there exist multipliers $\lambda \in I\!R^l$ and $\mu \in I\!R^m$

$$\begin{array}{l} \sum_{i=1}^{i=l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla g_j(\bar{x}) = 0, \\ \left(\sum_{i=1}^{i=l} \lambda_i \nabla^2 f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla^2 g_j(\bar{x}) \right)(y,y) \geqq 0, \\ \lambda \geq 0, \quad \mu \geqq 0, \quad \lambda_i = 0 \ \ \forall i \notin B(y), \quad \mu_j = 0 \ \ \forall j \notin E(y). \\ B(y) = \{\ i \in \{1,...,l\} \mid \nabla f_i(\bar{x})y = 0\ \} \\ E(y) = \{\ j \in \{1,...,m\} \mid g_j(\bar{x}) = 0, \quad \nabla g_j(\bar{x})y = 0\ \} \end{array}$$

Proof. Let y be a critical direction. Then, the system

$$\begin{array}{l} \nabla f_{B(y)}(\bar{x})z + \nabla^2 f_{B(y)}(\bar{x})(y,y) < 0, \\ \nabla g_{E(y)}(\bar{x})z + \nabla^2 g_{E(y)}(\bar{x})(y,y) \leqq 0. \end{array} \tag{10}$$

has no solution z. Which is equivalent to

$$\begin{array}{ll} \nabla f_{B(y)}(\bar{x})z + \nabla^2 f_{B(y)}(\bar{x})(y,y)t & < 0, \\ \nabla g_{E(y)}(\bar{x})z + \nabla^2 g_{E(y)}(\bar{x})(y,y)t & \leqq 0, \\ -t & < 0. \end{array}$$

has no solution $z \in I\!R^n,\ t \in R$.

By Motzkin's theorem of the alternative [7], there exist multipliers $\xi \in R,\ \lambda \in I\!R^l$ and $\mu \in I\!R^m$ such that

$$\begin{array}{l} \sum_{i=1}^{i=l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla g_j(\bar{x}) = 0, \\ \left(\sum_{i=1}^{i=l} \lambda_i \nabla^2 f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla^2 g_j(\bar{x}) \right)(y,y) - \xi = 0, \\ (\lambda, \xi) \geq 0, \quad \mu \geqq 0, \quad \lambda_i = 0 \ \ \forall i \notin B(y), \quad \mu_j = 0 \ \ \forall j \notin E(y). \end{array}$$

Since $(\lambda, \xi) \geq 0$ implies ($\lambda \geq 0$ and $\xi \geqq 0$) or ($\lambda \geqq 0$ and $\xi > 0$), hence, there exist multipliers $\lambda \in I\!R^l$ and $\mu \in I\!R^m$ such that either (11) or (12) holds:

$$\begin{array}{l} \sum_{i=1}^{i=l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla g_j(\bar{x}) = 0, \\ \left(\sum_{i=1}^{i=l} \lambda_i \nabla^2 f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla^2 g_j(\bar{x}) \right)(y,y) > 0, \\ \lambda \geqq 0, \quad \mu \geqq 0, \quad \lambda_i = 0 \ \ \forall i \notin B(y), \quad \mu_j = 0 \ \ \forall j \notin E(y). \end{array} \tag{11}$$

$$\begin{array}{l} \sum_{i=1}^{i=l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla g_j(\bar{x}) = 0, \\ \left(\sum_{i=1}^{i=l} \lambda_i \nabla^2 f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla^2 g_j(\bar{x}) \right)(y,y) \geqq 0, \\ \lambda \geq 0, \quad \mu \geqq 0, \quad \lambda_i = 0 \ \ \forall i \notin B(y), \quad \mu_j = 0 \ \ \forall j \notin E(y). \end{array} \tag{12}$$

Let us assume that (12) does not hold. Which is equivalent to the inconsistency of the system

$$\begin{array}{l} \sum_{i=1}^{i=l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla g_j(\bar{x}) = 0, \\ \left(\sum_{i=1}^{i=l} \lambda_i \nabla^2 f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla^2 g_j(\bar{x}) \right)(y,y) - \xi = 0, \\ \lambda \geq 0, \quad \xi \geqq 0, \quad \mu \geqq 0, \quad \lambda_i = 0 \ \ \forall i \notin B(y), \quad \mu_j = 0 \ \ \forall j \notin E(y). \end{array}$$

By Motzkin's theorem of the alternative [7], there exist z and $t \geqq 0$ satisfying

$$\nabla f_{B(y)}(\bar{x})z + \nabla^2 f_{B(y)}(\bar{x})(y,y)t < 0,$$
$$\nabla g_{E(y)}(\bar{x})z + \nabla^2 g_{E(y)}(\bar{x})(y,y)t \leqq 0.$$

Since (10) has no solution, we have $t = 0$; hence,

$$\nabla f_{B(y)}(\bar{x})z < 0, \quad \nabla g_{E(y)}(\bar{x})z \leqq 0.$$

On the other hand,

$$\nabla f_{B(y)}(\bar{x})y = 0, \quad \nabla f_{L\setminus B(y)}(\bar{x})y < 0,$$
$$\nabla g_{E(y)}(\bar{x})y = 0, \quad \nabla g_{E\setminus E(y)}(\bar{x})y < 0.$$

because y is critical. Thus, it holds that

$$\nabla f(\bar{x})(y + \epsilon z) < 0, \quad \nabla g_E(\bar{x})(y + \epsilon z) \leqq 0.$$

for any sufficiently small $\epsilon > 0$, which contradicts the first-order necessary conditions for efficiency. This completes the proof. □

Now we turn to discuss second-order sufficient conditions.

4 Sufficient Conditions for Efficiency

Theorem 4.1: Suppose that any f_i, g_j are quasiconvex and twice continuously differentiable at $\bar{x} \in A$. If for each critical direction $y \neq 0$, there exist $\lambda \in \mathbb{R}^l$ and $\mu \in \mathbb{R}^m$ such that

$$\sum_{i=1}^{i=l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla g_j(\bar{x}) = 0, \tag{13}$$

$$\left(\sum_{i=1}^{i=l} \lambda_i \nabla^2 f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla^2 g_j(\bar{x}) \right)(y,y) > 0, \tag{14}$$

$$\lambda \geq 0, \quad \mu \geqq 0, \quad \lambda_i = 0 \ \ \forall i \notin B(y), \quad \mu_j = 0 \ \ \forall j \notin E(y). \tag{15}$$

Then, $\bar{x}$ is an efficient solution to problem (P).

Proof. Assume that for each critical direction $y \neq 0$, there exist $\lambda \in \mathbb{R}^l$, and $\mu \in \mathbb{R}^m$ such that (13) - (15) hold, but $\bar{x}$ was not efficient solution to problem (P). Then, there is $x \in A$ such that

$$f(x) \leq f(\bar{x}) \tag{16}$$

From the quasi-convexity of f and g and (16) we obtaint

$$\nabla f(\bar{x})(x-\bar{x}) \leqq 0,$$
$$\nabla g_E(\bar{x})(x-\bar{x}) \leqq 0.$$

We distinguish two cases:

• if $\nabla f(\bar{x})(x-\bar{x}) < 0$, for $d = x-\bar{x}$, the following system

$$\nabla f(\bar{x})d < 0,$$
$$\nabla g_E(\bar{x})d \leqq 0.$$

is inconsistent, by Motzkin's theorem of the alternative the following system

$$\lambda \nabla f(\bar{x}) + \mu \nabla g(\bar{x}) = 0,$$
$$\lambda \geq 0, \quad \mu \geqq 0, \quad \mu_j = 0 \ \ \forall j \notin E$$

is inconsistent, which contradicts (13) and (14).

• if $\nabla f_r(\bar{x})(x-\bar{x}) = 0$, for at least one $r \in \{1,...,l\}$, then, $d = x-\bar{x}$ is a non zero critical direction.

Take $x(t) = \bar{x} + td, \quad t \in]0,1]$

From the quasi-convexity of f, we have:

$$f(\bar{x}+td) - f(\bar{x}) = t\nabla f(\bar{x})d + \frac{t^2}{2}\nabla^2 f(\bar{x})(d,d) + o(t^2) \leqq 0.$$

Hence,

$$\nabla f(\bar{x})d + t/2\Big(\nabla^2 f(\bar{x})(d,d) + o(t^2)/t^2\Big) \leqq 0 \tag{17}$$

Similary,

$$\nabla g_E(\bar{x})d + t/2\Big(\nabla^2 g_E(\bar{x})(d,d) + o(t^2)/t^2\Big) \leqq 0. \tag{18}$$

By assumption, there exist $\lambda \in I\!R^l$ and $\mu \in I\!R^m$ such that (13) - (15) hold. Multiplying (17) and (18) with λ and μ respectively, we summarize to get

$$\sum_{i=1}^{i=l} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla g_j(\bar{x})$$
$$+t/2\Big\{\Big(\sum_{i=1}^{i=l} \lambda_i \nabla^2 f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla^2 g_j(\bar{x})\Big)(d,d) + o(t^2)/t^2\Big\} \leqq 0$$

Noting expression (13) and $t > 0$, we obtain

$$\Big(\sum_{i=1}^{i=l} \lambda_i \nabla^2 f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla^2 g_j(\bar{x})\Big)(d,d) + o(t^2)/t^2 \leqq 0.$$

Using expression (15) again and $t \longrightarrow 0^+$, we get

$$\left(\sum_{i=1}^{i=l} \lambda_i \nabla^2 f_i(\bar{x}) + \sum_{j=1}^{j=m} \mu_j \nabla^2 g_j(\bar{x})\right)(d,d) \leqq 0$$

which contradicts (14). Therefore, $\bar{x}$ is an efficient solution to problem (P). □

References

[1] Aghezzaf, B., "Second-Order Necessary Conditions of Kuhn-Tucker type in Multiobjective Optimization Problems", In: *XIIth International Conference on Multiobjective Criteria Decicion Making*, Fernuniversität Hagen, Germany, June 19 -23, 1995

[2] Aghezzaf, B. and Hachimi, M., "Second-Order Optimality Conditions in Multiobjective Programming Problems", In: *Conférence Internationale en Recherche Opérationnelle*, Université Cadi Ayyad, Marrakech, Maroc, Juin 25-27, 1996.

[3] Ben-Tal, A., "Second-Order and Related Extremality Conditions in Nonlinear Programming", *Journal of Optimization Theory and Applications*, Vol. 31, 1980, pp. 143 - 165

[4] Kawasaki, H., "Second-order Necessary Conditions of The Kuhn-Tucker type under New Constraint Qualification", *Journal of Optimization Theory and Applications*, Vol. 57, 1988, pp. 253 - 264.

[5] Lin, J. G., "Maximal Vectors and Multi-objective Optimization", *Journal of Optimization Theory and Applications*, Vol. 18, 1967, pp. 41 - 64.

[6] Maeda, T., "Constraint Qualification in Multiobjective Optimization Problems : Differentiable Case", *Journal of Optimization Theory and Applications*, Vol. 80, 1994, pp. 483 - 500.

[7] Mangasarian, O. L., *Nonlinear Programming*, McGrow Hill, New York, New York, 1969.

[8] Marusciac, I., "On Fritz John type Optimality Criterion in Multiobjective Optimization", *L'Analyse Numérique et la Theorie de l'Approximation*, Vol. 11, 1982, pp. 109 - 114.

[9] Singh, C., "Optimality Conditions in Multiobjective Differentiable Programming", *Journal of Optimization Theory and Applications*, Vol. 53, 1987, pp. 115 - 123.

[10] Wang, S., "Second-Order Necessary and Sufficient Conditions in Multiobjective Programming", *Numer. Funct. Anal. and Optimiz*, Vol. 12, 1991, pp. 237 - 252.