Proper Efficiency in Nonconvex Vector-Maximization-Problems with Polyhedral Domination Structure

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Abstract

Vector-maximization-problems arise when more than one objective function are to be simultaneously maximized over a feasibility region. The concept of proper efficiency has been introduced by Geoffrion in order to eliminate points of a certain anomalous type.In multicriteria decision problems, the trade-offs play an important role and are related to the weighted sum programs. When the trade-offs belong to an interval, we obtain a polyhedral structure of dominance. In this paper, a generalization is provided for the characterization of the properly efficient solutions, as solutions of some parametric programmig problems.

Keywords: Vector-Maximization-problems, trade-offs, properly efficient solutions, domination structures, weighted sum programs.

1 Introduction

Given k $(k \geq 2)$ criterion functions $f_1, \ldots, f_k$, to be simultaneously maximized on a given feasibility region $X \subset \mathbb{R}^n$ and denoting by $f(x)$ the vector function $(f_1(x), \ldots, f_k(x))$ for $x \in X$. The vector-Maximization-problem (VMP)

$$\underset{x \in X}{Max}\; f(x)$$

is defined (following Geoffrion [4], for instance) as the problem of finding all efficient points in X; $x^* \in X$ is efficient (or $z^* = f(x^*)$ is Pareto optimal) if there is no $x \in X$ such that $f_i(x) \geq f_i(x^*)$ for each i and $f_j(x) > f_j(x^*)$ for at last one j; in other terms, x^* is efficient if, for all $x \in X$, $f(x) - f(x^*)$ belongs to the non negative orthant of $\mathbb{R}^k$, then $f(x) = f(x^*)$.

The VMP is related to the family $(P_\lambda)$of scalar maximum problem and this play an important role in multiobjective optimization:

$$(P_\lambda) \quad \underset{s.t.\ x\in X}{Max} \quad (\lambda.f(x))$$

where $\lambda \in \overline{\Lambda}^+ = \left\{(\lambda_1,\ldots,\lambda_k) \in \mathbb{R}^k \,/\, \sum_{i=1}^{i=k} \lambda_i = 1;\ \lambda_i \geq 0 \text{ for each } i\right\}$ and $(\,.\,)$ denotes the usual inner product in $\mathbb{R}^k$.

Let us denote Λ^+ the set of $\lambda \in \overline{\Lambda}^+$ with all $\lambda_i > 0$. It is well- known that if X is an intersection of hyperplanes and the f_i linear then $x^* \in X$ optimal in (P_λ) for some $\lambda \in \Lambda^+$ if and only if x^* is efficient (see e.g.[9]).

In [4], under the hypothesis that X is convex and all f_i concave, Geoffrion characterizes the solutions of scalar problems (P_λ), $\lambda \in \Lambda^+$, as the properly efficient solutions. For the aider convenience, let us recall that $x^* \in X$ is a properly efficient solution if x^* is efficient and there exists a scalar $M > 0$ such that for each i,

$$\frac{f_i(x) - f_i(x^*)}{f_j(x^*) - f_j(x)} \leq M,$$

for some j such that $f_j(x) < f_j(x^*)$, whenever $x \in X$ and $f_i(x) > f_i(x^*)$. The improperly efficient solutions are undesirable solutions because at such point, the marginal gain on at least one criterion can be made arbitrary large with respect to each of the marginal losses in other criteria.

The need for relaxing Geoffrion's hypothesis arose from the consideration of very important practical cases where the functions f_i are not concave. So it's no wonder that the question was revisited by several authors: (see [2], [3], [5], [6] and [11]).

In multicriteria problems,the trade-offs play an important role. They are used to interpret the compensation between the loss on a criterion and the gain on another one: it is the quantity that the decision maker permits to sacrify on a reference criterion to compensate the gain of one unit on another one. The trade-offs are not known previously, and the different decision makers have to express their point of view and make propositions. One can expect that trade-offs vary in an interval. Let us assume that for all $j \neq k$, the trade-offs s_j verify:

$0 < m_j \leq s_j \leq M_j$ and $m_j < M_j$ for all $j \neq k$ and $s_k = 1$, where f_k is the reference criterion.

The s_j are related to the weights λ_j of the criteria f_j and we have:

$s_j = \frac{\lambda_j}{\lambda_k}$ for all j.
and the admissible weights belongs to the set

$$D^* = \left\{\lambda \in \mathbb{R}^k : \text{ for all } j \neq k,\ \lambda_j - \lambda_k M_j \leq 0 \text{ and } \lambda_k m_j - \lambda_j \leq 0\right\}.$$

Following Yu [12], the domination cone D associated with our multicriteria problem is defined by

$$D = \left\{ v \in \mathbb{R}^k : \ Av \leq 0 \right\},$$

where A is a $m \times k$ matrix with $m = 2^{k-1}$ and the ith component of each row vector u^j of A has the following form (see Tamura [10]):

$$u_i^j = \begin{cases} m_i \text{ or } M_i \text{ if } i \neq k \\ 1 \qquad\qquad \text{if } i = k \end{cases}$$

So, for $x,\ y \in X$, the decision maker prefers x to y if and only if $f(y) \in f(x) + D$ and $f(y) \neq f(x)$.

Let $Z = \{z = f(x),\ x \in X\} \subset \mathbb{R}^k$

Definition 1: $z^* \in Z$ is D-nondominated (or D_Pareto optimal) if and only if $(z^* - D) \cap F = \{z^*\}$, where $z^* - D = \{z^* - d,\ d \in D\}$. $x^* \in X$ such that $z^* = f(x^*)$ z^* D-nondominated is called D-efficient. Following [12], we denote $EXT\,[Z \mid D]$ the set of the D-nondominated solutions.

One can easily verify that $\ker A = \emptyset$ and applying lemma 2.3.4. (p.31) in [8] we have: $z^* \in EXT\,[Z \mid D]$ if and only if $Az^* \in EXT\left[AZ \mid \mathbb{R}^m_-\right]$.

Remarks

1. D^* is the polar cone of D and the rows of the matrix A are the generators of D^* (see Tamura [10]).

2. Each D-efficient solution $x \in X$ for the initial problem is an efficient solution for the (VMP) above where the informations given by the domination structure D are introduced in the criteria and conversely.

3. The number of criteria is more important in the (VMP) above, than the original problem and these new criteria cannot have a real significance for the decision maker. But this situation can turn in good advantage to the analyst in well understanding the preferences of the decision maker.

Definition 2: $x^* \in X$ is D-properly efficient if and only if x^* is properly efficient in the (VMP):

$$\underset{x \in X}{Max} \ \ Af(x)$$

Otherwise, x^* is D-efficient if there exist a scalar $M > 0$ such that for each i,

$$(u^i.(f(x) - f(x^*))) \leq M(u^j.(f(x^*) - f(x)))$$

for some j such that $(u^j.(f(x^*)-f(x))) > 0$,whenever $x \in X$ and $(u^i.(f(x)-f(x^*))) > 0$.

2 Main Result

Let consider the vector maximization problem defined in section 1 with the trade-offs s_j in the interval $[m_j, M_j]$ and without any assumption concerning X or the f_i's. We work in the criterion space $\mathbb{R}^k$ where the set $X \subset \mathbb{R}^n$ is mapped by the criterion vector function $f = (f_1, \ldots, f_k)$; $Z \subset \mathbb{R}^k$ denotes the image set $\{f(x),\ x \in X\}$.

Let $Z^{\leq}$ denote the convex hull of $(Z + D)$ and $(P_\lambda)_Z$ the problem of finding $Max\,\{(\lambda.z)\,,\ z \in Z\}$. Note that for $z^* \in Z$ and $\lambda \in D^*$ we have z^* is optimal in $(P_\lambda)_Z$ if and only if z^* is optimal in $(P_\lambda)_{Z^{\leq}}$.

Lemma 1: For any D-nondominated point $z^* \in Z$, the following conditions are equivalent:

(a) z^* is optimal in $(P_\lambda)_Z$ for some $\lambda \in D^*$.

(b) z^* lies on the boundary of $Z^{\leq}$.

Proof. If z^* satisfies (a), then there exists $\lambda \in D^*$ such that for all $z \in Z$; we have

$$(\lambda.z) \leq (\lambda.z^*)$$

Then, for all $z \in Z$ and all $d \in D$ we have:

$$(\lambda.(z + d)) \leq (\lambda.z) \leq (\lambda.z^*).$$

Thus , for all $z \in Z^{\leq}$, $(\lambda.z) \leq (\lambda.z^*)$ and z^* lies necessary in the boundary of $Z^{\leq}$.

Conversely, z^* satisfies (b), referring to corollary 11.6.2. in [7], there exist $\lambda \in \mathbb{R}^k$ such that for all $z \in Z^{\leq}$, $(\lambda.z) \leq (\lambda.z^*)$.

In particular, for all $z \in Z$ and for all $d \in D$, we have

$$(\lambda.(z + d)) \leq (\lambda.z) \leq (\lambda.z^*).$$

So that $\lambda \in D^*$ and z^* is optimal in $(P_\lambda)_Z$. ■

A straightforward geometric interpretation of (a) is, the existence of a supporting hyperplane $\{z : (\lambda.(z - z^*)) \leq 0\}$ in z^* with $\lambda \in D^*$ and for all $z \in Z$, $(\lambda.z) \leq (\lambda.z^*)$; z^* is called D-supported.

Let us transpose the notion of proper efficiency to the criterion space:

$z^* \in Z$ is said to be properly D-nondominated in Z if $z^* \in EXT\,[Z \mid D]$ and there exists a scalar $M > 0$ such that, for $z \in Z$ and each i with $(u^i.z) > (u^i.z^*)$ we have:

$$(u^i.(z - z^*)) \leq M(u^j.(z^* - z)),$$

for some j such that $(u^j.(z^* - z)) > 0$.

$z^* \in Z$ is said properly D-nondominated in $Z^{\leq}$, if we replace Z by $Z^{\leq}$ in this definition.

The fundamental results characterizing proper D-efficient solutions in terms of the solutions of $(P_\lambda)_Z$ are given in theorems 4 and 5.

Theorem 2: if $z^* \in Z$ is optimal in $(P_\lambda)_Z$ for some $\lambda \in intD^*$, then z^* is properly D-nondominated in Z.

Proof. Let Z be optimal in $(P_\lambda)_Z$ for some $\lambda \in intD^*$. For all $z \in Z$ we have:

$$(\lambda.(z - z^*)) \leq 0$$

Let us show that $z^* \in EXT\,[Z \mid D]$. Suppose to the contrary that there exist $z \in Z$ and $d \in D - \{0\}$ such that $z = z^* - d$. Then, we have

$$(\lambda.\,(-d)) > 0.$$

Thus $(\lambda.z) = (\lambda.z^*) + (\lambda.\,(-d)) > (\lambda.z^*)$.

This contradicts the fact that z^* is optimal in $(P_\lambda)_Z$. So $z^* \in EXT\,[Z \mid D]$.

Let us show that z^* is properly $D-$nondominated in Z. Let $\lambda \in D^*$ and $u^1, \ldots, u^m$ the generators of D^*: there exists $\alpha_1, \ldots, \alpha_m > 0$ such that $\lambda = \sum_{i=1}^{i=m} \alpha_i u^i$.

Let $M = (m-1) \underset{i,j}{Max} \dfrac{\alpha_j}{\alpha_i}$

Suppose to the contrary that for some i and some $z \in Z$, we have:

$(u^i.(z - z^*)) > M(u^j.(z^* - z))$ for all $j \neq i$ such that $(u^i.(z - z^*)) > 0$.

Then for all $j \neq i$ we have:

$$(u^i.(z - z^*)) > \frac{m-1}{\alpha_i}\alpha_j(u^j.(z^* - z)).$$

Multiplying by $\dfrac{\alpha_i}{m-1}$ and summing on $j \neq i$ we obtain:

$$\alpha_i(u^i.(z - z^*)) > \sum_{j \neq i} \alpha_j(u^j.(z^* - z)).$$

Consequently we have: $(\lambda.(z - z^*)) > 0$. This contradicts the optimality of z^* in $(P_\lambda)_Z$. ■

The converse of theorem 4 is valid under some convexity assumptions as given in theorem 5.

Theorem 3: If X is convex and the f_i's are concave then $z^* \in Z$ is properly D-nondominated if and only if z^* is optimal in $(P_\lambda)_Z$ with $\lambda \in intD^*$.

Proof. The necessary condition is given by theorem 4.

Recall that $z^* \in Z$ is properly D-nondominated if and only if Az^* is properly nondominated in the $(VMP) \left\{ \underset{z \in Z}{Max}\ Az \right.$.

By theorem 2 in [4], this is equivalent to:
there exists $\alpha \in int\mathbb{R}^m_+$ such that

$$(\alpha, (A(z - z^*))) = (A^T\alpha, (z - z^*)) \leq 0.$$

Taking $\lambda = A^T\alpha$, λ is a positive combination of the m generators of D^*. Thus $\lambda \in intD^*$. ■

Theorem 4 gives an extension of Geoffrion theorem 2 in [4] to the case where the domination structure is given by a polyhedral cone.

For $\lambda = \sum_{i=1}^{i=m} \lambda_i u^i \in D^*$ and $z \in Z$, consider the following notations:
$I(\lambda) = \{i : \lambda_i = 0\}$; $I^c(\lambda) = \{i : \lambda_i > 0\}$; $I_z(\lambda) = \left\{i \in I(\lambda) : (u^i.(z - z^*)) > 0\right\}$ and $|I|$ denotes the cardinality of the set I.

For a given $z^* \in Z$, consider the following conditions:

(a) z^* is optimal in $(P_\lambda)_Z$ for some $\lambda \in intD^*$.

(b) z^* is properly D-nondominated $Z^{\leq}$

(c) There exist a $\lambda^* \in D^*$ such that z^* is optimal in $(P_{\lambda^*})_Z$ and there exist a real number $M > 0$ such that, for each $z \in Z$ and each $i \in I_z(\lambda^*)$ we have:

$$(u^i.(z - z^*)) \leq M \sum_{j \in I^c(\lambda^*)} (u^j.(z^* - z)) \tag{1}$$

(d) z^* is properly D-nondominated in Z.

The following theorem gives an extension of theorem 5 where no convexity conditions are assumed.

Theorem 4: If $z^* \in EXT[Z \mid D]$, then the conditions (a), (b) and (c) are equivalent.

Proof. (b)$\Rightarrow$(a). Let $z^* \in Z$ be properly D-nondominated in $Z^{\leq}$. The real functions defined on $\mathbb{R}^k$ by $g_i(z) = z_i - z_i^*$ for $i \in \{1, \ldots, k\}$, are concave (linear) on $Z^{\leq}$; and $Z^{\leq}$ is convex . Let us consider the (VMP) :$\underset{z \in Z^{\leq}}{Max}\ g(z)$,

where $g(z) = (g_1(z), \ldots, g_k(z))$. It is obvious that z^* is properly D-nondominated in (VMP) above. Thus, by theorem 5 we have for some $\lambda \in intD^*$ and for each $z \in Z^{\leq}$,

$$(\lambda, g(z)) \leq (\lambda, g(z^*)) = 0.$$

This means that z^* is optimal in $(P_\lambda)_{Z^{\leq}}$ with $\lambda \in intD^*$ and so, it is optimal in $(P_\lambda)_Z$ for some $\lambda \in intD^*$ $\left(Z \subseteq Z^{\leq}\right)$.

(a)$\Rightarrow$(b). Let z^* be optimal in $(P_\lambda)_Z$ for some $\lambda \in intD^*$. Then z^* is optimal in $(P_\lambda)_{Z^{\leq}}$. Thus, by theorem 4, z^* is properly D-nondominated in $Z^{\leq}$ (cf. remark above).

(a)$\Rightarrow$(c). Suppose that z^* is optimal in $(P_\lambda)_Z$ for some $\lambda^* \in intD^*$. For $\lambda = \lambda^*$, we have $I(\lambda^*) = \emptyset$ and for each real number $M > 0$, the relation (1) in (c) holds.

(c)$\Rightarrow$(a).Suppose that z^* is optimal in $(P_{\lambda^*})_Z$ for some $\lambda^* \in D^*$ and there exists a real number $M > 0$ such that (1) holds for each $z \in Z$ and for each $i \in I_z(\lambda^*)$;then we get, for all $z \in Z$ the following inequalities:

$$\begin{aligned}
\sum_{i \in I(\lambda^*)} (u^i.(z - z^*)) &\leq \sum_{i \in I_z(\lambda^*)} (u^i.(z - z^*)) \\
&\leq -M\,|I_z(\lambda^*)| \sum_{i \in I^c(\lambda^*)} (u^i.(z - z^*)) \\
&\leq -M\,|I(\lambda^*)| \sum_{i \in I^c(\lambda^*)} (u^i.(z - z^*))
\end{aligned}$$

Hence

$$\sum_{i \in I(\lambda^*)} (u^i.(z - z^*)) + M\,|I_z(\lambda^*)| \sum_{i \in I^c(\lambda^*)} (u^i.(z - z^*)) \leq 0$$

So, z^* is optimal in $(P_\lambda)_Z$ with $\lambda^* \in intD^*$ defined as follow:

$$\lambda = \sum_{i=1}^{i=m} \lambda_i u^i \text{ with } \lambda_i = \left\{ \begin{array}{c} 1 \text{ if } i \in I(\lambda^*) \\ M\,|I(\lambda^*)| \text{ if } i \in I^c(\lambda^*) \end{array} \right.$$

With theorem 6 we obtain a generalisation of theorem 2 in [1] to the case where the domination structure is defined by a polyhedral cone and this occurs when the trade-offs belongs to an interval.

Corollary 5: If X is convex and the f_i's are concave; the conditions (a), (b) and (c) are equivalent to (d).

So, theorem 5 characterizes the properly D-nondominated solutions in $Z^{\leq}$ but does not apply without convexity assumptions: Theorem 6 provides a characterization of the properly D-nondominated solutions in $Z^{\leq}$ without any assumption.

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