On Modelling Preventive Replacements of Randomly Deteriorating Systems*

Robert Cléroux

Département de Mathématiques et de Statistique
Université de Montréal

Abstract
In this paper we describe the preventive replacement model for randomly deteriorating single unit systems and illustrate some of the main difficulties met in practical situations. We discuss in more details two special cases: a replacement model with a mixture of actions at failure and a nonparametric replacement model based on the bootstrap methodology. It is seen that although the general replacement problem might be relatively simply formulated, obtaining the objective function as much explicitly as possible is not always a trivial matter and finding a numerically efficient algorithm to obtain the optimal solution is not always easy.

1 The Preventive Replacement Problem

A system is operating until failure and its life length is a random variable. At failure it is repaired or replaced by a similar equipment. A failure is often expensive and/or dangerous and therefore one looks for preventive actions before failure, but as late as possible before failure. Here a system is any piece of complex equipment such as an airplane, an electricity distribution network, a nuclear system or a production system. It can also be any simple unit such as a transistor or a lightbulb.

The optimal replacement strategy, in a given class of strategies, is the one optimizing a given objective function (cost function, availability, reliability, etc.). Thus a preventive replacement problem is defined by three elements: (i) a class of policies or strategies amongst which we look for the optimal one, (ii) a cost structure and (iii) an optimality criterion.

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1.1 Classes of Stragegies

Many authors have proposed a classification of the preventive replacement strategies that are found in the scientific litterature. See for example McCall (1965), Pierskalla and Voelker (1976), Sherif an Smith (1981) and Valdez Flores and Feldman (1989). Dekker (1995) proposed an integrating framework wich covers many maintenance activities. Here we consider only two general practical situations which in fact define two general classes of replacement policies: (i) it is possible to monitor, at reasonable costs, the evolution of the system in real time; (ii) it is not possible to do so for several reasons: high costs, too many systems to be monitored, etc.

In the first situation, the preventive replacement strategies are based on the age of the system in the following way: at age T (optimization parameter), the equipment is preventively replaced and if a failure occurs before age T, some corrective action is undertaken. This action can consist of a minimal repair which restores the failure rate of the system to its value just prior to failure, an imperfect repair, that is a minimal repair which is successfull with probability p and unsuccessfull with probability $1-p$, an inactivity period, a repair if its random cost is smaller than a given threshold, a temporary replacement by a used unit (a working unit which has been removed after age T), or a mixture of these actions. Each choice of action defines a specific mathematical model and consequently a specific class of replacement strategies.

But monitoring the system may not be feasible for practical reasons such as high costs involved, difficult access of some equipment or other technical constriants. When monitoring the system is not feasible, the replacement strategies are no longer based on the age of the system but rather on systematic periods of time: the equipment is preventively replaced at time $T, 2T, 3T, \ldots$ (optimization parameter) independently of its age or history, and at failure, we undertake one of the preceeding actions or any mixture of them. Again, each choice of action defines a specific mathematical model and a specific class of replacement policies.

1.2 Cost Structures

The objective function to be optimized is usually a cost function which depends on the corective and preventive actions. It depends explicitly on a fixed cost associated with preventive replacements, a fixed cost associated with replacement at failures, costs associated with system operation (either a sequence of discrete costs or a continuous and increasing cost function), random repair costs which could depend on the age of the system, costs associated with using a used unit, costs associated with inactivity, and so on. These costs can be expressed in time units if we are interested in maximizing system's availibility of in monitary units if we are interested in minimizing average total expenses. The objective function also depends on the life distributions of the new or used equipment and on other functions related to the renewal process.

1.3 Optimality Criterion

The optimization problem is to find the policy parameters which minimize, under all the constraints of the model, the average total cost per time unit over an infinite horizon. Some other criteria are however found in the litterature. They relate to finite horizons or to maximizing the system's availability. In the examples below, the first criterion will be used.

1.4 Major Practical Difficulties

The major practical difficulties met when solving a preventive replacement problem are (i) to obtain the objective function as much explicitly as possible and (ii) to find a numerically efficient algorithm to obtain the optimal solution. Moreover, when the underlying system's life distribution is unknown or only partially known some computer intensive methods may be needed. The examples below will illustrate the level of difficulty we often meet in practical problems.

1.5 Solution Methodology

In general there are two approches to solve such a problem in the continuous time framework: a macro approach and a micro approach. In the macro approach one usually makes use of renewal theory and optimal stopping theory and their limit theorems. The average total cost per time unit over an infinite time span is expressed as the ratio of the average cost during the renewal cycle over the average length of the cycle. A global analysis of the renewal cycle is made. The micro approach is based on the marginal cost of deferring maintenance. We proceed to a marginal cost analysis: the marginal cost function is the expected additional cost incurred when, at a given instant of preventive replacement, the replacement is deferred for an infinitesimal period Δt. This cost is noted $\eta(t)$. From $\eta(t)$ one can obtain $C(t)$, the average total cost per time unit over an infinite time span. Then the optimal solution of the preventive replacement problem is a solution of $\eta(t) = C(t)$.

Both approaches have been widely used in practice. The second approach is simpler to use but leads, when both approaches can be used, to an objective function which is less explicit than that obtained from the macro approach which is based on a more elaborate mathematical machinery. This marginal cost analysis has been introduced by Berg (1980) and successfully used by Berg and Cléroux (1982 a,b) and Berg, Bienvenu and Cléroux (1986). A complete account of this approach is given in Berg (1995). The macro approach is illustrated in the example below.

2 A First Example: the Block Replacement Problem with Multiple Choice at Failure

This problem has been considered in Ait Kadi and Cléroux (1988) and it is briefly summarized here as an illustration.

The equipment is preventively replaced by a new one at systematic times and at failure it is either replaced by a new or a used equipment or remains inactive until the next planned replacement. The decision at failure is made according to the time interval in which the failure occurs, in the following way: (i) preventive replacements by new equipments are made at times kT, $k = 1, 2, \ldots$ independently of the equipment's failure history, (ii) if a failure occurs in the time interval $[(k-1)T, kT - \delta_1)$, $k = 1, 2, \ldots\ ;\ 0 \leq \delta_1 \leq T < \infty$, the equipment is replaced by a new one, (iii) if it occurs in the time interval $[kT - \delta_1, kT - \delta_2]$, $k = 1, 2, \ldots$, $0 \leq \delta_2 \leq \delta_1 \leq T < \infty$, the equipment is replaced by a used one; (iv) if it occurs in $[kT - \delta_2, kT)$, $k = 1, 2, \ldots$, the equipment remains inactive or works less efficiently until the next planned replacement at time kT.

A used equipment is an equipment which has been removed in a planned replacement and which is still working. It has age T. The replacement strategy is completely specified by (T, δ_1, δ_2) and these parameters will be found by minimizing the average total cost of the policy, per time unit, over an infinite time span.

The following assumptions are required in what follows: (1) the successive life lengths are iid according to some known increasing failure rate (IFR) distribution, (2) failures are instantly detected, (3) preventive replacements and replacements at failure are made instantly, (4) replacements are made perfectly and do not affect the equipments characteristics, (5) the failure state is an absorbing state, (6) the numbers of new and used equipments available for replacement is sufficient to avoid shortages, (7) a used item costs less than a new one, (8) an item cannot be installed more than twice, and (9) the preventive replacement cost is less than the cost of replacement at failure and both costs are known.

In order to write down the mathematical model, some notation is needed:

$F(t)$: distribution function of successive life lengths with density function $f(t)$ and failure rate $r(t)$;

$M(t)$: renewal function corresponding to $F(t)$, with renewal density $m(t)$;

$N_1(t)$: number of failure replacements with new equipments in the time interval $[0, t]$;

$N_2(t)$: number of preventive replacements (with new equipments) in $[0,t]$;

$N_3(t)$: number of failure replacements with used equipments in $[0,t]$;

D : length of inactivity period;

c_1 : cost when replacing at failure with new equipment;

c_2 : cost when replacing preventively;

c_3 : cost when replacing at failure with used equipment;

c_4 : cost associated with each time unit of inactivity;

$f_T(t) = f(t+T)/[1-F(T)] =$ density function of the lifetime of a used item with distribution function $F_T(t)$, renewal function $M_T(t)$ and renewal density $m_T(t)$.

It is natural to assume $c_2 < c_1$ and $c_1 - c_2 \leq c_3 \leq c_1$. The problem is to find $(T^*, \delta_1^*, \delta_2^*)$ which minimizes

$$C(T,\delta_1,\delta_2) = \lim_{t\to\infty}\left\{c_1\frac{E[N_1(t)]}{t} + c_2\frac{E[N_2(t)]}{t} + c_3\frac{E[N_3(t)]}{t} + c_4\frac{E[D]}{t}\right\} \tag{1}$$

subject to $T-\delta_1 \geq 0$, $\delta_1 - \delta_2 \geq 0$, $T \geq 0$, $\delta_1 \geq 0$, $\delta_2 \geq 0$. First we must write down Equation (1) as much explicitly as possible. In this case, this is done using essentially renewal theory and its limit theorems. We know from renewal theory that

$$\lim_{t\to\infty}\frac{E[N_1(t)]}{t} = \frac{M(T-\delta_1)}{T} \tag{2}$$

and that

$$\lim_{t\to\infty}\frac{E[N_2(t)]}{t} = \frac{1}{T} \tag{3}$$

and we obtain (see Ait Kadi and Cléroux (1988)),

$$\begin{aligned}\lim_{t\to\infty}\frac{E[N_3(t)]}{t} &= \frac{1}{T}\int_{T-\delta_1}^{T-\delta_2}\big[1+M_T(T-\delta_2-x)\big]f(x)dx \\ &+ \frac{1}{T}\int_0^{T-\delta_1}\int_{T-\delta_1}^{T-\delta_2}\big[1+M_T(T-\delta_2-x)\big]m(y)f(x-y)dxdy \end{aligned} \tag{4}$$

and

$$\begin{aligned}
\lim_{t\to\infty}\frac{E[D]}{t} &= \frac{1}{T}\int_{T-\delta_2}^{T}(T-x)f(x)dx\\
&+\frac{1}{T}\int_{T-\delta_2}^{T}\int_{0}^{T-\delta_1}(T-x)m(y)f(x-y)dydx\\
&+\frac{1}{T}\int_{T-\delta_2}^{T}\int_{T-\delta_1}^{T-\delta_2}(T-x)f(v)f_T(x-v)dvdx\\
&+\frac{1}{T}\int_{T-\delta_2}^{T}\int_{T-\delta_1}^{T-\delta_2}\int_{0}^{T-\delta_1}(T-x)m(y)f(v-y)f_T(x-v)dydvdx\\
&+\frac{1}{T}\int_{T-\delta_2}^{T}\int_{T-\delta_1}^{T-\delta_2}\int_{0}^{T-\delta_2-v}(T-x)f(v)m_T(v+u)f_T(x-u-v)dudvdx\\
&+\frac{1}{T}\int_{T-\delta_2}^{T}\int_{T-\delta_1}^{T-\delta_2}\int_{0}^{T-\delta_2-v}\int_{0}^{T-\delta_1}(T-x)m(y)\\
&\qquad f(v-y)m_T(v+u)f_T(x-u-v)dydudvdx. \qquad (5)
\end{aligned}$$

The problem is a nonlinear optimization problem with linear constraints. It can be solved using the generalized reduced gradient algorithm found in Lasdon, Fox and Ratner (1974). This algorithm solves a series of reduced problems using the method of Broyden, Fletcher and Shanno (see Goldfarb (1969) or Fletcher (1970) for example). The stopping rule is based on the Karuch-Kuhn-Tucker (KKT) conditions.

The above equations involve $M(t)$, $M_T(t)$, $m(t)$ and $m_T(t)$ where $M(t) = \sum_{n=1}^{\infty} F^{(n)}(t)$ and where $F^{(n)}(t)$ is the n^{th} convolution of $F(t)$ with itself (and similarly for $M_T(t)$, $m(t)$ and $m_T(t)$). These functions cannot be written explicitly, except for a few special cases, and must therefore be computed numerically. In order to do this we can use the algorithm introduced in Cléroux and McConalogue (1976) which use a cubic spline approximation of $F(t)$ and $F_T(t)$ and a Lobatto quadrature formula to obtain $F^{(n)}(t)$, $F_T^{(n)}(t)$, $M(t)$ and $M_T(t)$. The functions $m(t)$ and $m_T(t)$ are computed in a similar fashion. These computations are made a large number of times in the algorithm which solves the optimization problem. More on the computation of functions arising in renewal theory can be found in McConalogue (1978, 1981) and Baxter et al (1981, 1982).

The algorithm requires an initial feasible solution (T, δ_1, δ_2). Many trials have to be carried out. In some cases, changing the initial solution results in a change in the solution. However the optimal cost does not change greatly.

Many numerical results are given in Ait Kadi and Cléroux (1988) and comparisons are made with many other replacement policies which are special cases of the policy presented here. Let us just mention a simple numerical example to end this section.

Let $F(t) = 1 - e^{-\lambda t^{\alpha}}$, the Weibull distribution with parameters $\alpha > 0$ and $\lambda > 0$ and $t \geq 0$. Suppose $\alpha = 2$ and $\lambda = 0.0001$ so that the mean and the standard deviation of $F(t)$ are respectively $\mu = 88.62$ and $\sigma = 46.33$ and for the cost structure, let $\frac{c_2}{c_1} = 0.10$, $\frac{c_1-c_3}{c_2} = 0.03$ and $\frac{c_4}{c_1} = 0.01$. The above numerical procedure has been used with several initial solutions. The optimal solution is given by $T^* = \delta_1^* = \delta_2^* = 56.58$ with $C(T^*, \delta_1^*, \delta_2^*) = 27.39 \times 10^{-4}$. This corresponds to the strategy: "preventive replacements are planned at intstant kT, $k = 1, 2, \ldots$ with $T = 56.58$, and if a failure occurs, the system remains inactive until the next planned replacement". Numerical comparisons are made with other policies in Ait Kadi and Cléroux (1988).

3 Nonparametric Models

In the above model and in most replacement models found in the scientific litterature it is assumed that $F(t)$ is completely known. However, in many practical situations, the distribution $F(t)$ is unknown or some parameters of $F(t)$ are unknown. In these cases, the optimal solution of the model together with the optimal cost of the policy have to be estimated.

When $F(t)$ is unknown, two approaches are possible starting with a random sample $X_1, X_2, \ldots, X_n$ obtained experimentally from $F(t)$ and with the empirical distribution function (e.d.f.) $F_n(t)$.

3.1 Classical Nonparametric Approach

This approach consists of solving the preventive replacement problem using $F_n(t)$ in place of $F(t)$. The procedure is the following:

Step 1 : In the cost function noted $C(T, F)$ where T is the vector of parameters defining the replacement policy, replace $F(t)$ by $F_n(t)$ to obtain $C(T, F_n)$.

Step 2 : Minimize $C(T, F_n)$ with respect to T to obtain $T_0(F_n)$ and $C(T_0(F_n), F_n)$.

Then $T_0(F_n)$ is an estimate of $T_0(F)$ and $C(T_0(F_n), F_n)$ is an estimate of $C(T_0(F), F)$. Now the properties of these estimators have to be established and in particular, we must investigate their convergence to the true values.

This approach has been used by Arunkumar (1972) in a nonparametric setup and by Ingram and Scheaffer (1976) in a parametric setup with unknown parameters for the simple age replacement model. Sequential estimators of $T_0(F)$ have been obtained by Bather (1977) and Frees and Ruppert (1985) again for the age replacement model.

3.2 Bootstrap Procedure

The bootstrap is a computer intensive procedure (see Efron (1979), Efron and Tibshirani (1986) for example) which can be used here to obtain confidence intervals for the optimal replacement policy and the optimal cost. It has been used in Léger and Cléroux (1992) to obtain a confidence interval for the optimal cost of the age replacement problem. The procedure is as follows:

Step 1 : Obtain $T_0(F_n)$ and $C(T(F_n), F_n)$ as above.

Step 2 : Choose a random sample $Y_1, Y_2, \ldots, Y_n$, with replacement, from $X_1, X_2, \ldots,$ X_n and compute

a) the corresponding e.d.f. $G_n(t)$ together with $T_0(G_n)$ and $C(T_0(G_n), G_n)$.

b) $v = C(T_0(G_n), G_n) - C(T_0(G_n), F_n)$.

Step 3 : Repeat Step 2 B times and obtain the sampling distribution of v. Let $v_{.05}$ and $v_{.95}$ be its 5% and 95% centiles respectively.

Step 4 : Obtain the 90% bootstrap confidence interval for $C(T_0(F_n, F)$ given by $[C(T_0\ (F_n)\ ,F_n) - v_{.95},\ C(T_0(F_n),\ F_n) - v_{.05}]$ called the bootstrap pivotal confidence interval.

Step 5 : Repeat Steps 2 to 4 K times and compute the empirical coverage probability.

One can also obtain the bootstrap t confidence interval which consists of bootstrapping a properly normalized version of v (see Léger and Cléroux (1992)). Some theoritical developments have to be made to obtain the normalizing factor and to show that the bootstrap confidence intervals have a coverage probability which converges to the nominal confidence coefficient.

4 A Second Example: Bootstrap Solution of the Age Replacement Problem

Consider the simple age replacement policy : the equipment is replaced at failure or at age T, whichever occurs first. Let c_1 be the constant cost of replacement at failure and $c_2 < c_1$ the constant cost of a preventive replacement.

Then it is known from renewal theory that

$$C(T, F) = \frac{c_1 F(T) + c_2[1 - F(T)]}{\int_0^T [1 - F(u)]du}. \tag{6}$$

Tab. 1: Solution of the Age Replacement Problem

$F(t)$	Gamma	Truncated Normal	Weibull
$T_0(F)$	4036	4127	3923
$C(T_0(F), F)$	0.030	0.037	0.037

Tab. 2: Sample of size $n = 50$

		Empirical coverage probability		Nominal
$F(t)$	$C(T_0(F), F)$	Bootstrap pivotal	Bootstrap t	Confidence coefficient
Gamma	0.030	.9090	.8924	.90
Truncated Normal	0.037	.8926	.9106	.90
Weibull	0.037	.8742	.8842	.90

The optimal policy $T_0(F)$ must be a solution of $\frac{\partial C}{\partial T} = 0$ or of

$$r(T) \int_0^T [1 - F(u)]du - F(T) = \frac{c_2}{c_1 - c_2} \tag{7}$$

where $r(T) = f(T)/[1 - F(T)]$ is the failure rate of $F(T)$. If $r(T)$ is continuous and increasing to ∞, the solution $T_0(F)$ exists and is unique and we get

$$C(T_0(F), F) = (c_1 - c_2)r(T_0(F)). \tag{8}$$

As a numerical example we discuss a replacement problem related to electron tubes used in airline communications. Here we have $\mu = 9080$ and $\sigma = 3027$ for the underlying life distribution $F(t)$ where the time unit is the hour. We also have $c_1 = 1100$ and $c_2 = 100$. This example has been considered by Glasser (1967) for the age replacement model and by Hanscom and Cléroux (1975) for the block replacement model. Table 1 gives the solution obtained by Glasser when $F(t)$ is gamma, truncated normal or Weibull.

Léger and Cléroux (1992) have used the bootstrap methodology in these cases, where $T_0(F)$ and $C(T_0(F), F)$ can be computed, in order to study its accuracy numerically. They used $n = 20, 50$; $B = 1000$; $K = 5000$. Some results for the bootstrap pivotal and bootstrap-t are given in Table 2. For more results the reader is referred to Léger and Cléroux (1992).

These results together with those found in Léger and Cléroux (1992) suggest that the bootstrap methodology may be used with success in other preventive replacement problems as well.

5 Conclusion

In this paper we introduced the main elements defining a preventive replacement problem and considered two examples to illustrate how to obtain the objective function as much explicitly as possible, how to obtain the numerical solution of the problem and how to approach such a problem in a nonparametric context. There are, of course, many more interesting preventive replacement problems in the scientific litterature which deal, for instance, with shocks or multivariate cases and which could not be considered here. The interested reader is refered to Abdel-Hameed (1986), Sheu and Griffith (1992), Block and Savits (1988), Shaked and Shantikumar (1986) and the references therein.

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