Outputs from a Loss System with Two Stations and a Smart (Cyclic) Server*

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Abstract

We study the interoutput and interdeparture distributions from a loss system with one server attending two stations. In this system, after a completion of service, the server will attend the customer waiting for service in the other station (if any) or the first customer that arrives in either station. We assume independence among arrivals and service. The arrival at each station is a Poisson process and the service distribution is exponential. We model this system using Markov renewal processes embedded at output and departure times. Using these structures and filtering techniques, we determine the interoutput distribution from one of the stations. The total departure process, consisting of the outputs and overflow streams is also considered. Conditions for this to be a Poisson process are found. Numerical and simulation results are also presented.

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1 Introduction

We consider a loss system with one server shared between two stations. The server is "smart" in the sense that if, on completion of service at one station, it sees a customer present at the other station it will switch over to that station. If both stations are empty the server will switch (if necessary) to serve the first arriving customer. The arrival processes are Poisson and the service times at each station are exponential,

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and independent of each other. From the point of view of one station, this system can be seen as a vacation model, where the vacation corresponds to the time the server spends at the other station. The system we consider is also an example of a polling system with zero switchover times. The interest in vacation models and polling systems comes from their application to computer/communications systems which include several interconnected stations. In these cases, the departures from one station will be the input to a subsequent station.

In Doshi [5], a survey of vacation models is presented. Several works on vacations are concern with queue length and waiting time distributions, for instance, Kao and Narayanan [7], Takagi [13] and Browne and Kella [1]. In Tedijanto [15] stochastic comparisons for the departure, queue size and waiting times processes are made for two service policies in multiple-vacations models. Kleinrock and Levy [9] describe random polling systems with zero switchover times. Since we have only two stations our system is actually also a cyclic polling system as presented in Takagi [14]. Other models of polling include different policies of service and non zero switching and walking times, see for example, Ibe [6] and Srinivasan, Niu and Cooper [11]. To the best of our knowledge there are not many studies on the departure process from this kind of system. Magalhães, McNickle and Salles [10] is concerned with the departure process in a similar model whose server switches according to a policy depending on the last station served and the state of the system just after a departure. The interdeparture distribution from either of the stations is computed and it is compared with the one step projection approximation. Stanford and Fisher [12] consider a system with one station, two types of interarrival distributions and service in order of arrival, independent of the types. They computed the Laplace-Stieltjes transform and the coefficient of variation of the interdeparture distribution for one of the streams of arrivals. Looking just at one type of arrival, the time that the server is busy with the other type, can be viewed as a vacation. We expect our paper to motivate other studies in more general models.

The model and the continuous time stationary distribution are presented in sections 2 and 3. Section 4 has the basic results of the Markov renewal process for the output process and the inter-output distribution from station 1. Section 5 gives some numerical results for this process. The departure process which consists of all the served or overflowing customers is considered in Section 6. Simulation results are presented on section 7.

2 The Model

We suppose that we have two stations, 1 and 2, and one server. Customers arrive to station 1 or 2 in different flows. At any time, there can be no more than one customer at each station. Customers that are denied entry to the system, overflow and are lost. When the server completes a service time at any station, it will switch to serve the next available customer. This may be a customer already present at the other station, or if the system is empty the server will wait for, and switch to, the first arrival.

We assume that the arrival processes at each station are independent Poisson processes, with rates $\lambda_i,\ i = 1, 2$. The service time at station $i,\ i = 1, 2$, has an exponential distribution with rate μ_i, independent of the arrival process.

3 The Steady State Distribution

In order to determine the throughput we first determine the steady state continuous-time distribution. The possible states of the system are $(1,0,0)$, $(1,1,0)$, $(1,1,1)$, $(2,0,0)$, $(2,0,1)$, and $(2,1,1)$. Here (i,j,k) means that the server is at station i, and the number of customers present is given by the vector (j,k). Note that the states $(1,0,1)$ and $(2,1,0)$ have zero probability as if when they occur the server instantaneously switches to $(2,0,1)$ and $(1,1,0)$ respectively. The balance equations are:

$$
\begin{aligned}
(\lambda_1 + \lambda_2)p_{100} &= \mu_1 p_{110} \\
(\lambda_2 + \mu_1)p_{110} &= \lambda_1 p_{100} + \lambda_1 p_{200} + \mu_2 p_{211} \\
\mu_1 p_{111} &= \lambda_2 p_{110} \\
(\lambda_1 + \lambda_2)p_{200} &= \mu_2 p_{201} \\
(\lambda_1 + \mu_2)p_{201} &= \lambda_2 p_{200} + \lambda_2 p_{100} + \mu_1 p_{111} \\
\mu_2 p_{211} &= \lambda_1 p_{201}
\end{aligned}
$$

These equations do not have a particularly neat form for the solution. With the assistance of the symbolic algebra package, MAPLE [2], a typical term is:

$$
\begin{aligned}
p_{100} = (\mu_1^2\mu_2\lambda_1(\lambda_2 + \lambda_1 + \mu_2))/((\lambda_1 + \lambda_2)(\mu_2\lambda_2^2\mu_1 + \mu_2\lambda_2^2\lambda_1 + \lambda_2^2\lambda_1\mu_1 + \\
\mu_1^2\lambda_1\lambda_2 + \lambda_1^2\mu_1\lambda_2 + \mu_2^2\mu_1\lambda_2 + \mu_2\mu_1^2\lambda_2 + \mu_2\lambda_1^2\lambda_2 + \mu_2^2\lambda_2\lambda_1 + \\
2\lambda_2\mu_2\mu_1\lambda_1 + \mu_2\mu_1^2\lambda_1 + \mu_2^2\mu_1\lambda_1 + \mu_2^2\mu_1^2 + \mu_2\lambda_1^2\mu_1))
\end{aligned}
$$

Since the arrival process at station 1 is Poisson, the PASTA result is valid and the probability that an arriving customer is permitted to enter the service facility is $p_{100} + p_{200} + p_{201}$. Thus the throughput at station 1 is:

$$
\lambda_1(p_{100} + p_{200} + p_{201}) \tag{1}
$$

4 The Markov Renewal Process Embedded at Output Times

We consider the system just after an output (service completion) from station 1 or station 2 and represent these instants by $\boldsymbol{T}$. We also assume that the server has completed any switching between stations. It is straightforward to verify that the process $(\boldsymbol{X}, \boldsymbol{N_1}, \boldsymbol{N_2}, \boldsymbol{T})$, where $\boldsymbol{X}$ gives the location of the server, and $\boldsymbol{N_1}, \boldsymbol{N_2}$ is the number of customers left behind by the leaving customer, is a Markov renewal process with states $\{(100), (201), (200), (110)\}$. We take the states to be in that order, so the first two states constitute departures from station 1, and the last two are departures from station 2. The kernel $QO(t)$ of the Markov renewal process is given by:

$$QO(t) = \begin{bmatrix} J(t,\mu_1,\lambda_1,\lambda_2) & K(t,\mu_1,\lambda_1,\lambda_2) & J(t,\mu_2,\lambda_2,\lambda_1) & K(t,\mu_2,\lambda_2,\lambda_1) \\ 0 & 0 & G(t,\mu_2,\lambda_1) & H(t,\mu_2,\lambda_1) \\ J(t,\mu_1,\lambda_1,\lambda_2) & K(t,\mu_1,\lambda_1,\lambda_2) & J(t,\mu_2,\lambda_2,\lambda_1) & K(t,\mu_2,\lambda_2,\lambda_1) \\ G(t,\mu_1,\lambda_2) & H(t,\mu_1,\lambda_2) & 0 & 0 \end{bmatrix}$$

Here

$$G(t,\mu,a) = \int_{s=0}^{t} \mu e^{-\mu s} e^{-as} ds, \quad H(t,\mu,a) = \int_{s=0}^{t} \mu e^{-\mu s}(1-e^{-as}) ds,$$

$$J(t,\mu,b,a) = \int_{s=0}^{t} b e^{-(a+b)s} G(t-s,\mu,a) ds,$$

$$K(t,\mu,b,a) = \int_{s=0}^{t} b e^{-(a+b)s} H(t-s,\mu,a) ds$$

The Laplace-Stieltjes transform of the semi-Markov kernel is

$$QS(s) =$$

$$\begin{bmatrix} \frac{\lambda_1\mu_1}{(s+\lambda_2+\mu_1)(s+\lambda_2+\lambda_1)} & \frac{\lambda_1\lambda_2\mu_1}{(s+\mu_1)(s+\lambda_2+\mu_1)(s+\lambda_2+\lambda_1)} & \frac{\lambda_2\mu_2}{(s+\lambda_1+\mu_2)(s+\lambda_2+\lambda_1)} & \frac{\lambda_1\lambda_2\mu_2}{(s+\mu_2)(s+\lambda_2+\mu_2)(s+\lambda_2+\lambda_1)} \\ 0 & 0 & \frac{\mu_2}{s+\lambda_1+\mu_2} & \frac{\mu_2\lambda_1}{(s+\lambda_1+\mu_2)(s+\mu_2)} \\ \frac{\lambda_1\mu_1}{(s+\lambda_2+\mu_1)(s+\lambda_2+\lambda_1)} & \frac{\lambda_1\lambda_2\mu_1}{(s+\mu_1)(s+\lambda_2+\mu_1)(s+\lambda_2+\lambda_1)} & \frac{\lambda_2\mu_2}{(s+\lambda_1+\mu_2)(s+\lambda_2+\lambda_1)} & \frac{\lambda_1\lambda_2\mu_2}{(s+\mu_2)(s+\lambda_2+\mu_2)(s+\lambda_2+\lambda_1)} \\ \frac{\mu_1}{s+\lambda_2+\mu_1} & \frac{\mu_1\lambda_2}{(s+\lambda_2+\mu_1)(s+\mu_1)} & 0 & 0 \end{bmatrix}$$

The one-step transition matrix of the underlying Markov chain $\{\boldsymbol{X}, \boldsymbol{N_1}, \boldsymbol{N_2}\}$ is $QS(0)$, which gives the following steady-state distribution:

$$\pi = \left(\frac{\lambda_1\mu_1(\lambda_1+\lambda_2+\mu_2)}{S}, \frac{\lambda_1\lambda_2(\lambda_1+\lambda_2+\mu_2)}{S}, \frac{\lambda_2\mu_2(\lambda_1+\lambda_2+\mu_1)}{S}, \frac{\lambda_1\lambda_2(\lambda_1+\lambda_2+\mu_1)}{S} \right)$$

Here S is the sum of all the numerator terms. Note that this is not the same as the steady-state distribution as we are dealing with different processes. Since the first two states of the Markov renewal process correspond to outputs from station 1 we partition the $QS(s)$ matrix into four 2×2 matrices of the form:

$$QS(s) = \begin{bmatrix} B_{11}(s) & B_{12}(s) \\ B_{21}(s) & B_{22}(s) \end{bmatrix}$$

By the usual filtering techniques (Çinlar, [3]), the Laplace-Stieltjes transform of the time between returns to states corresponding to outputs from station 1 is:

$$Q_1(s) = B_{11}(s) + B_{12}(s)(I - B_{22}(s))^{-1}B_{21}(s).$$

A sample path argument shows that the stationary distribution for $Q_1(0)$ (the embedded Markov chain) is just a re-normalization of the equilibrium distribution by the total probability of visiting station 1. Representing it by $\phi = (\phi_{00}, \phi_{01})$, we have:

$$\phi_{00} = \pi_{100}/(\pi_{100} + \pi_{201})$$

$$\phi_{01} = \pi_{201}/(\pi_{100} + \pi_{201}),$$

where $\pi = (\pi_{100}, \pi_{201}, \pi_{200}, \pi_{110})$ is the equilibrium distribution of the embedded Markov chain $QO(\infty)$. The interoutput distribution from station 1, $q_1(t)$ can now be computed from inverting the product $\phi Q_1(s)\mathbf{e}$, where $\boldsymbol{e}$ is a column vector of 1's. Again the resulting expression is too complex to be worth writing down, but easily within the capability of MAPLE. The mean of the inter-output time from station 1 (i.e. the mean of $q_1(t)$) is found from:

$$-\frac{\partial(\phi Q_1(s)e)}{\partial s}|_{s=0}$$

This provided a useful verification of the whole calculation, by comparing the reciprocal of this with the expression for the throughput of station 1, (equation (1), from the balance equations). They were found to be identical analytical expressions.

We also wish to calculate the serial correlation (autocorrelation) between successive output intervals from station 1. If Y_n and Y_{n+1} are two successive interoutput times then in steady state:

$$E(Y_n, Y_{n+1}) = \frac{\partial^2(\phi Q_1(s_1)Q_1(s_2)e)}{\partial s_1 \partial s_2}|_{s_1=0, s_2=0}$$

Expressions for $E(Y_n^2)$ and hence $Var(Y)$ follow similarly.

5 Some Numerical Results for the Output Process

(a) $\lambda_1 = 1$, $\lambda_2 = 2$, and $\mu_1 = 3$, $\mu_2 = 4$

The interoutput distribution from station 1 is

$$q_1(t) = 1 - \frac{1}{60}e^{-7t} - \frac{5}{3}e^{-t} + \frac{19}{20}e^{-3t} - \frac{4}{15}e^{-4t}, \quad \textit{with mean} = 149/105$$

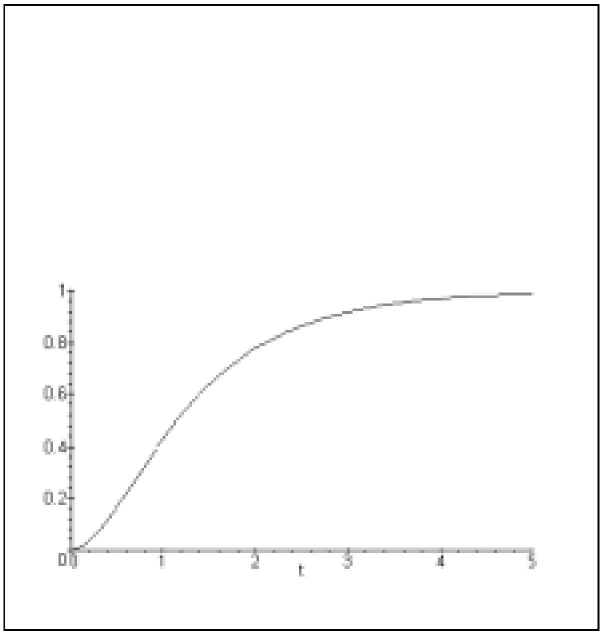

Fig. 1: Interoutput distribution from station 1 $\lambda_1 = 1$, $\lambda_2 = 2$, and $\mu_1 = 3$, $\mu_2 = 4$

The autocorrelation of the output intervals from station 1, calculated by the method outlined at the end of section 4, is $9/3599 = .0025$.

(b) $\lambda_1 = 2$, $\lambda_2 = 1$, and $\mu_1 = 4$, $\mu_2 = 3$

This is the complement of the situation (a), so it corresponds to the output from station 2 with the parameter settings of (a). The interoutput distribution from station 1 is:

$$q_1(t) = 1 + \frac{1}{15}e^{-6t} + \frac{32}{15}e^{-3t} - 3e^{-2t} - \frac{1}{5}e^{-4t}, \quad with\ mean = 149/180$$

Figure 1 (and Figure 2 below) look remarkably like two-phase distributions of the kind that result from the output of a single telephone line with no storage, i.e. an Erlang-B situation. We decided to see how close an approximation this would be. For a single server with no storage and an arrival rate of $\lambda_1 = 2$, the service rate to produce a throughput of $180/149$ can be found from the Erlang-B formula. We need to solve for μ in $(1 - B(1,a))\lambda_1 = 180/149$, where $B(1,a) = (\lambda_1/\mu)/(1 + (\lambda_1/\mu))$. The solution is $\mu = 180/59$. The resulting distribution of the times between outputs (which we shall call the "equivalent" Erlang-B) is:

$$q_{erl}(t) = 1 - \frac{90}{31}e^{-2t} + \frac{59}{31}e^{-\frac{180}{59}t}$$

This is plotted in Figure 2 along with $q_1(t)$. The two graphs are almost indistinguishable. Figure 3 plots the difference between $q_1(t)$ and $q_{erl}(t)$. The largest difference we have observed in any example is 0.01.

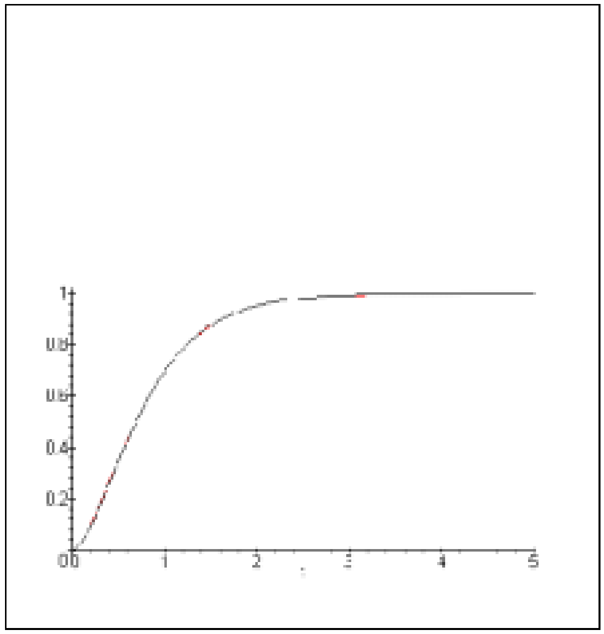

Fig. 2: $q_1(t)$ and $q_{erl}(t)$ for station 1 $\lambda_1 = 2$, $\lambda_2 = 1$, and $\mu_1 = 4$, $\mu_2 = 3$

The autocorrelation between successive interoutput intervals is $144/11729 = .013$. These small autocorrelations lead us to suggest that a renewal process, based on the "equivalent" Erlang-B, may well be an adequate approximation for the output process from a single station.

The switching server is surprisingly efficient here. If we had two separate servers with rates $m_1 = 3$ and $m_2 = 4$, the output rates from station 1 and from station 2 would be 0.75 and 1.33 customers per time unit respectively. The rates in our model, of $105/149 = .705$ and $180/149 = 1.208$, achieved with only one switching server, compare quite well with these.

6 The Markov Renewal Process Embedded at Departure Times

The departure times are the combination of outputs and overflow times. Let $T = \{T_n : n = 0, 1, ..\}$ represent these departure instants. For a number of systems (for example an $M/M/C/N$ queue) the departure process can be shown to be Poisson. We wish to determine if this is true for our system. The state of the system just after T_n is represented by (X, N_1, N_2, T), where X is the position of the server after the switchover (if any) and $\{N_1, N_2\}$ are the queue lengths at stations 1 and 2. The

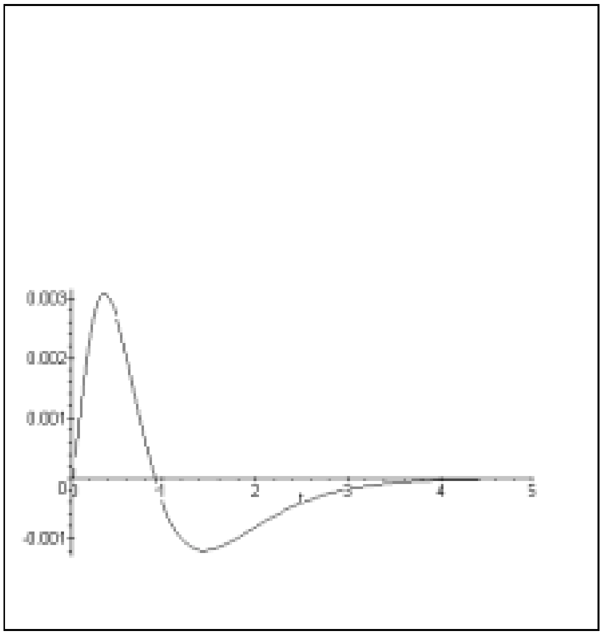

Fig. 3: The Difference Between $q_1(t)$ and the Erlang-B "Equivalent"

process (X, N_1, N_2, T) is now a Markov renewal process with states $\{(1,0,0), (2,0,0), (1,1,1), (2,0,1), (1,1,1), (2,1,1)\}$. The Laplace-Stieltjes transform of the kernel, $QD(s)$, is

$$\begin{bmatrix} \lambda_1\mu_1 ab_1 & \lambda_2\mu_2 ab_2 & \lambda_2\lambda_1\mu_2 ab_2^2 + \lambda_1^2 ab_1 & \lambda_1\lambda_2\mu_1 ab_1^2 + \lambda_2^2 ab_2 & (\lambda_1+\lambda_2)\lambda_1\lambda_2 ab_1^2 & (\lambda_1+\lambda_2)\lambda_1\lambda_2 ab_2^2 \\ \lambda_1\mu_1 ab_1 & \lambda_2\mu_2 ab_2 & \lambda_2\lambda_1\mu_2 ab_2^2 + \lambda_1^2 ab_1 & \lambda_1\lambda_2\mu_1 ab_1^2 + \lambda_2^2 ab_2 & (\lambda_1+\lambda_2)\lambda_1\lambda_2 ab_1^2 & (\lambda_1+\lambda_2)\lambda_1\lambda_2 ab_2^2 \\ \mu_1 b_1 & 0 & \lambda_1 b_1 & \lambda_2\mu_1 b_1^2 & (\lambda_1+\lambda_2)\lambda_2 b_1^2 & 0 \\ 0 & \mu_2 b_2 & \lambda_1\mu_2 b_2^2 & \lambda_2 b_2 & 0 & (\lambda_1+\lambda_2)\lambda_1 b_2^2 \\ 0 & 0 & 0 & \mu_1 b_1 & (\lambda_1+\lambda_2) b_1 & 0 \\ 0 & 0 & \mu_2 b_2 & 0 & 0 & (\lambda_1+\lambda_2) b_1 \end{bmatrix}$$

Here $a = 1/(\lambda_1 + \lambda_2 + s)$, and $b_i = 1/(\lambda_1 + \lambda_2 + \mu_i + s), i = 1, 2$. Note that by the way we have defined the state process, we do not distinguish between the overflow streams of station 1 and 2. This simplification is necessary to keep the number of states tractable.

From the transition matrix of the embedded Markov chain, $QD(0)$, the stationary distribution π_D was computed using MAPLE. Again it has complicated terms. It is interesting to note that although the state space is the same as that for the continuous time process considered in Section 3, the stationary distribution is not the same, as the queue-length process is not reversible for this system. Finally the distribution of an arbitrary interdeparture distribution was calculated from inverting $\pi_D QD(s).e$, where e is a column vector of ones. Again, checking that the mean of the interdeparture distribution is $1/(\lambda_1+\lambda_2)$ provides a useful verification of the calculation. The interdeparture distribution is given by

$$D(t) = 1 + \alpha_1 e^{-(\lambda_1+\lambda_2)t} + (\alpha_2 + \alpha_3 t)e^{-(\lambda_1+\lambda_2+\mu_1)t} + (\alpha_4 + \alpha_5 t)e^{-(\lambda_1+\lambda_2+\mu_2)t}$$

where the coefficients α_i can be written as β_i/γ_i, and

$$\gamma_1 = \mu_1\mu_2(\lambda_1^2\lambda_2\mu_2 + \lambda_1^2\mu_1\mu_2 + \lambda_1^2\mu_1\lambda_2 + \lambda_1\lambda_2^2\mu_2 + 2\lambda_1\mu_1\lambda_2\mu_2 + \lambda_1\mu_1\lambda_2^2 + \lambda_1\lambda_2\mu_2^2 + \lambda_1\lambda_2\mu_1^2 + \lambda_1\mu_1\mu_2^2 + \lambda_1\mu_2\mu_1^2 + \mu_2^2\mu_1^2 + \lambda_2\mu_1\mu_2^2 + \lambda_2\mu_2\mu_1^2 + \lambda_2^2\mu_1\mu_2),$$

$$\gamma_2 = \frac{\lambda_1+\lambda_2}{\mu_2}\gamma_1, \quad \gamma_3 = \frac{\lambda_1+\lambda_2}{\mu_1\mu_2}\gamma_1, \quad \gamma_4 = \frac{\lambda_1+\lambda_2}{\mu_1}\gamma_1, \quad \gamma_5 = \frac{\lambda_1+\lambda_2}{\mu_1\mu_2}\gamma_1,$$

$$\beta_1 = -(\mu_2^2\mu_1^2 + \lambda_1\lambda_2\mu_1^2 + \lambda_2\mu_2\mu_1^2 + \lambda_1\lambda_2\mu_2^2 + \lambda_1\mu_1\mu_2^2)(\lambda_1\mu_1 + \lambda_2\mu_2 + \mu_1\mu_2),$$

$$\beta_2 = -(\mu_1^2 + \lambda_1\mu_1 + \lambda_2\mu_1 + \lambda_1\lambda_2 + \lambda_2^2)(\mu_1 - \mu_2)\lambda_1\lambda_2\mu_2,$$

$$\beta_3 = -\lambda_1\lambda_2^2\mu_2(\lambda_1 + \lambda_2 + \mu_1)(\mu_1 - \mu_2),$$

$$\beta_4 = \lambda_2\lambda_1\mu_1(\mu_2^2 + \lambda_1\mu_2 + \lambda_2\mu_2 + \lambda_1^2 + \lambda_1\lambda_2)(\mu_1 - \mu_2),$$

$$\beta_5 = \lambda_1^2\mu_1\lambda_2(\lambda_1 + \lambda_2 + \mu_2)(\mu_1 - \mu_2)$$

Theorem: The departure process is a Poisson process with rate $\lambda_1+\lambda_2$ if, and only if, $\mu_1 = \mu_2$.

Proof:

1: Poisson departure process $\Rightarrow \mu_1 = \mu_2$.

If the departure is a Poisson process then the interdeparture distribution is exponential with parameter $\lambda_1+\lambda_2$. From the expressions for $D(t)$, this can only happen when $\mu_1 = \mu_2$.

2: $\mu_1 = \mu_2 \Rightarrow$ Poisson departure process.

With $\mu_1 = \mu_2 = \mu$,

$$\pi_D = \left[\frac{\lambda_1\mu^2}{(\lambda_1+\lambda_2)C} \; \frac{\lambda_2\mu^2}{(\lambda_1+\lambda_2)C} \; \frac{\lambda_1\mu}{C} \; \frac{\lambda_2\mu}{C} \; \frac{\lambda_1\lambda_2}{C} \; \frac{\lambda_1\lambda_2}{C}\right], \quad C = \lambda_1\mu + \lambda_2\mu + 2\lambda_1\lambda_2 + \mu^2$$

Using Maple, we have verified that

$$\pi_D Q D(t) = (1 - e^{-(\lambda_1+\lambda_2)t})\pi_D$$

and, from theorem 2.11.2 on page 47 of Disney & Kiessler [4], this is sufficient to imply that the departure process is a renewal process with a negative exponential distribution with parameter $\lambda_1 + \lambda_2$. This completes the proof.

When $\mu_1 \neq \mu_2$ the interdeparture distribution is still quite close to a negative exponential distribution.

Figure 4 plots the difference between $D(t)$ and a negative exponential distribution with parameter $\lambda_1 + \lambda_2 = 3$, for $\lambda_1 = 1$, $\lambda_2 = 2$, and $\mu_1 = 3$, $\mu_2 = 4$. The difference is now extremely small. In Section 7 we will use simulation to estimate the autocorrelation in the departure process. This is also found to be very small, so we suggest that a Poisson process approximation for the departure process may well be adequate.

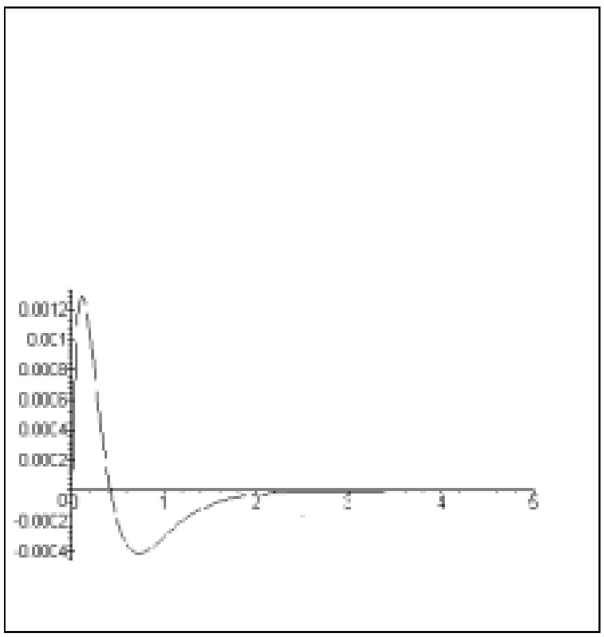

Fig. 4: The Difference Between $D(t)$ and Negative Exponential

7 Some Results from Simulation

We have a number of hypotheses which we have as yet been unable to confirm analytically. A simulation program for the system was written in GPSS/H to investigate these. The program is listed in the Appendix.

With $\lambda_1 = 1$, $\lambda_2 = 2$, $\mu_1 = 3$, $\mu_2 = 4$, and $50,000$ arrivals we get the following results:

The Departure Process

This is the superposition of the two output and the two overflow processes.

```
Descriptive Statistics
Variable        N        N*       Mean   Median   TrMean    StDev   SEMean
diff        49999         1    0.33489  0.23145  0.29742  0.33512  0.00150
Variable      Min       Max         Q1       Q3
diff      0.00000   3.34326    0.09521  0.46484
```

So the mean and standard deviation of the departure process are almost identical. Plotting the distribution of interdeparture intervals showed that it was statistically indistinguishable from a negative exponential distribution with mean $1/(\lambda_1+\lambda_2)$.The interdeparture intervals also are almost uncorrelated (below), hence we hypothesise that the departure process is actually very similar to a Poisson process, even for cases where $\mu_1 \neq \mu_2$.

```
Autocorrelation Function
ACF of diff
          -1.0 -0.8 -0.6 -0.4 -0.2  0.0  0.2  0.4  0.6  0.8  1.0
            +----+----+----+----+----+----+----+----+----+----+
  1  -0.004                         X
  2   0.002                         X
  3  -0.001                         X
  4  -0.002                         X
  5   0.005                         X
```

The Output Process from Station 1

```
Descriptive Statistics
Variable        N        N*      Mean   Median   TrMean    StDev   SEMean
dif1out     11684         1    1.4331   1.1670   1.3309   1.0766   0.0100
Variable      Min       Max        Q1       Q3
dif1out    0.0127   11.8682    0.6631   1.8984
```

So the output from station 1 has a mean greater than its standard deviation. The simulation mean is not significantly different from that calculated in section 5 $(149/105 = 1.42)$ and the sample autocorrelation (below) is not significantly different from the theoretical value of .0025.

```
Autocorrelation Function
ACF of dif1out

         -1.0 -0.8 -0.6 -0.4 -0.2  0.0  0.2  0.4  0.6  0.8  1.0
           +----+----+----+----+----+----+----+----+----+----+
  1  -0.005                         X
  2   0.004                         X
  3   0.008                         X
  4   0.001                         X
  5  -0.005                         X
```

The Overflow Process

This is the superposition of the overflow processes from the two stations.

```
Descriptive Statistics
Variable        N       N*      Mean   Median   TrMean    StDev   SEMean
dofover     18132        1    0.9234   0.4766   0.7745   1.1508   0.0085
Variable      Min      Max        Q1       Q3
dofover    0.0000  12.4277    0.1494   1.2766
```

The overflow has a standard deviation greater than its mean (i.e. a hyperexponential process) and appears to be weakly positively serially correlated (The acceptance region for the null hypothesis of no autocorrelation is $1.96/\sqrt{(18132)} = .014$)

```
Autocorrelation Function
ACF of dofover

         -1.0 -0.8 -0.6 -0.4 -0.2  0.0  0.2  0.4  0.6  0.8  1.0
           +----+----+----+----+----+----+----+----+----+----+
  1   0.019                         X
  2  -0.012                         X
  3   0.001                         X
  4  -0.005                         X
  5  -0.004                         X
```

8 Conclusions

Very little appears to be known about the outputs from polling or switching systems. In the case of a server who switches between two stations with limited storage we have characterized the output and departure processes. Some conclusions we can draw so far is that the output processes appear to be almost renewal, and can be efficiently modelled by formulating an "equivalent" Erlang-B system. We have shown that the departure process from the system is Poisson iff $\mu_1 = \mu_2$. So far we have found that even in other cases the departure process does not differ greatly from a Poisson process. These results suggest that a decomposition approach to networks involving this kind of system may be reasonable.

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APPENDIX

The GPSS/H Program

```
        REAL &L1,&M1,&L2,&M2
        INTEGER &N
        PUTPIC
0Enter Arrival rate 1
        GETLIST &L1
        PUTPIC
 Enter Arrival rate 2
        GETLIST &L2
        PUTPIC
0Enter service rate 1
        GETLIST &M1
        PUTPIC
0Enter service rate 2
        GETLIST &M2
        PUTPIC
0How many arrivals
        GETLIST &N
        SIMULATE
        GENERATE     RVEXPO(3,1/&L1)
        ADVANCE 0
        TEST E FU(SERV1),0,OVER1
        SEIZE SERV1
        GATE NU SERV2
        ADVANCE RVEXPO(4,1/&M1)
        RELEASE SERV1
        BPUTPIC FILE=MARCOS,(C1)
******.***** 1
        TERMINATE 1
        GENERATE RVEXPO(3,1/&L2)
        ADVANCE 0
        TEST E FU(SERV2),0,OVER1
        SEIZE SERV2
        GATE NU SERV1
        ADVANCE RVEXPO(4,1/&M2)
        RELEASE SERV2
        BPUTPIC FILE=MARCOS,(C1)
******.***** 2
        TERMINATE 1
OVER1   BPUTPIC FILE=MARCOS,(C1)
*****.****** 0
```

```
    TERMINATE 1
    START &N,NP
END
```