A Proof of Local Maximum Principle for Optimal Control Problems with Mixed State Constraints

Paul Bosch *Juan A. Gómez*

Centro de Matemática y Física Teórica del ICIMAF
Calle E No. 309, Vedado, Habana, Cuba.
pbosch@dim.uchile.cl, jagomez@cidet.icmf.inf.cu

Abstract

The local Pontryagin's maximum principle for smooth optimal control problems with mixed control-state constraints is proved. It is obtained from more general necessary optimality conditions, given by a version of the Karush-Kuhn-Tucker theorem in Banach spaces. A new condition is introduced, in order to identify the multiplier associated to mixed phase constraints with a measurable function.

Keywords: Optimal Control problems, Phase constraints, Karush-Kuhn-Tucker theorem, Dubovitskii-Milyutin approach, Local maximum principle.

1 Introduction

Karush-Kuhn-Tucker (KKT) theorems are first order necessary conditions for the local minimum of smooth nonlinear optimization problems. In finite dimensional case, they are systematically studied in classical books of Nonlinear Programming. In the infinite dimensional case some limited versions were given in the books of Luenberger [30] and Girsanov [13]. There are also many papers with generalized versions of KKT-theorem (see for example [6], [9],[10], [12], [15], [18], [25], [26], [34], [36], [39], [40] and [41]).

On the other hand, there are several proofs of Pontryagin's Maximum Principle, as a necessary condition for optimal control problems, using different general approaches. We can mention, for example, the works of Neustadt [36] and Ioffe-Tijomirov [18]. One of the more popular general optimization theories, the Dubovitskii-Milyutin approach, is described extensively in the book of Girsanov [13]. In present days there are extensions of the Maximum Principle even for differential inclusions in the non differentiable case (see for example [28] and [35]).

In a recent paper (see [9]) we presented a version of KKT theorem, for problems containing equality and inequality operator constraints, and we applied this theorem to optimal control problems without phase constraints, deriving the local Pontryagin Maximum Principle from the KKT necessary conditions. The present paper is an extension of the results given in [9], because phase constraints are also considered, and is based in the internal report [10].

The optimal control problems with phase constraints have been studied by many authors, (see for example [20], [32], [33], [42] and the extense bibliography given in [17]). The necessary conditions are well known and widely used like a recipe but, as is pointed out in the survey of Hartl, Sethi and Vickson [17], it seems that there is no a published proof of the general case. This is the case when there are two phase constraints, one depending on the control and state variables and the other one depending only on the state variable. The desired proof should include conditions for "regularizing" the multipliers, which means the existence of them as functions instead of regular measures. A recent example of this kind of results is the paper of Aseev [3], where sufficient conditions for the absence of singular component of the measure are given, although the optimal control problem with phase constraints is formulated in a differential inclusion context, where the control variable does not appear explicitly.

In this paper we take a step to fill this gap, giving a full proof of the *local* maximum principle, which is derived from the KKT theorem, and a new condition for regularizing the multiplier corresponding to the mixed phase constraint.

2 Karush Kuhn Tucker theorem

The general optimization problem can be written as follows:

$$\begin{array}{ll} \min & F(x) \\ s.t. & H(x) = \theta_{\mathrm{Y}} \\ & G(x) \leq \theta_{\mathrm{Z}} \\ & g_j(x) \leq 0 \quad j = 1, ..., n \\ & x \in \mathcal{X} \subset \mathrm{X} \end{array} \tag{1}$$

where $F, g_j : \mathrm{X} \longrightarrow \Re$ are functionals; $H : \mathrm{X} \longrightarrow \mathrm{Y}$, and $G : \mathrm{X} \longrightarrow \mathrm{Z}$, are operators; X, Y are Banach spaces and Z is a normed space, ordered by a cone P, which is convex, pointed, proper and satisfies $int(P) \neq \emptyset$. Furthermore, θ_{Y} and θ_{Z} denote the corresponding zeros of Y and Z and $\mathcal{X}$ is a convex subset of X.

We shall use the following general version of Karush-Kuhn-Tucker (KKT) theorem:

Theorem 1 (Karush-Kuhn-Tucker): **:** Let x_0 be a local minimum of the problem (1) and suppose F, G, g_j $(j = \overline{1,n})$ are Frechet differentiable and H continuously Frechet differentiable in a neighborhood V of x_0, and $\mathcal{X}$ is a convex set with non empty interior in X. Suppose, in addition, that $DH(x_0)(\mathrm{X})$ is a closed subspace of Y and there is $\overline{h} \in \mathrm{X}$ such that:

$$G(x_0) + DG(x_0)(\overline{h}) \in -(int(P)) . \tag{2}$$

Then, there exist linear functionals $y^* \in \mathrm{Y}'$, $z^* \in \mathrm{Z}'$ and multipliers $\lambda_0, \mu_j \in \Re$, $(j = \overline{1,n})$, not all zeros, satisfying:

$$\lambda_0 DF(x_0) + y^* \circ DH(x_0) + z^* \circ DG(x_0) + \sum_{j=1}^{n} \mu_j Dg_j(x_0) = f, \tag{3}$$

where $z^* \geq \theta_{\mathrm{Z}'}$, $z^*[G(x_0)] = 0$, $\mu_j \geq 0$, $\mu_j g_j(x_0) = 0$, $(j = \overline{1,n})$ and f is a support functional of $\mathcal{X}$ at x_0. Moreover, if $DH(x_0)$ is surjective and $\bar{h}$ also satisfies $Dg_j(x_0)(\overline{h}) < 0$, for all $j \in J_0(x_0)$, $\overline{h} = \zeta(x - x_0)$, for $\zeta > 0$, $x \in int\mathcal{X}$, and $DH(x_0)(\overline{h}) = \theta_{\mathrm{Y}}$ then, it can be taken $\lambda_0 = 1$ in (3).

Some versions of this theorem are proven in [6], [34] and [40]. The Gateaux differentiability can be used instead of the Frechet one but then, a generalization of Lyusternik theorem is needed. Ledzewicz-Kowalewska [26] had shown how to do it, using results of Altman [1], but only for problems without inequality operator constraints.

In the next section we give the landmarks of a proof, based in Dubovitskii-Milyutin approach. An extension of this general theory is given in [27], where the Local Maximum Principle was also obtained, but it was applied to a simpler optimal control problem which does not have inequality operator constraints. Full details of the following proof can be found in [10].

3 Proof of the KKT theorem

We denote by:

$$Q_1 = \{x \in \mathrm{X} \mid H(x) = \theta_{\mathrm{Y}}\} ,$$
$$Q_2 = \{x \in \mathrm{X} \mid G(x) \leq \theta_{\mathrm{Z}}\} ,$$
$$Q_3 = \bigcap_{j=1}^{n} Q_{3j} = \bigcap_{j=1}^{n} \{x \in \mathrm{X} \mid g_j(x) \leq 0\} ,$$
$$Q_4 = \{x \in \mathrm{X} \mid x \in \mathcal{X}\} ,$$

the sets associated with operator and functional equality, inequality and inclusion constraints.

Initially, some particular cases can be excluded:

1. If $DF(x_0) = \Theta_{\mathcal{L}(\mathrm{X},\mathrm{Y})}$, we take $\lambda_0 = 1$, $z^* = \theta_{\mathrm{Z}'}$, $y^* = \theta_{\mathrm{Y}'}$, $\mu_j = 0, j = \overline{1,n}$, and $f = \theta_{\mathrm{X}'}$.

2. If $DH(x_0)$ is not a surjective operator then, by Hanh-Banach's theorem, there exists $y^* \in \mathrm{Y}'$, $y^* \neq \theta_{\mathrm{Y}'}$ such that $y^* \circ DH(x_0) = \theta_{\mathrm{Y}}$, since $DH(x_0)(\mathrm{X})$ is a closed subspace in Y. Therefore, we can take $\lambda_0 = 0$, $z^* = \theta_{\mathrm{Z}'}$, $\mu_j = 0, j = \overline{1,n}$, and $f = \theta_{\mathrm{X}'}$.

3. If $Dg_{j_0}(x_0) = \theta_{\mathrm{X}'}$, for some $j_0 \in J_0$, where:
$$J_0 = J_0(x_0) = \{j \in \{1,...,n\} \mid g_j(x_0) = 0\ \},$$
we take $\mu_{j_0} = 1$ and $\lambda_0 = 0$, $z^* = \theta_{\mathrm{Z}'}$, $y^* = \theta_{\mathrm{Y}'}$, $\mu_j = 0, j \neq j_0$, and $f = \theta_{\mathrm{X}'}$.

Hence, the theorem is true for these special cases and we may assume that $DF(x_0) \neq \Theta_{\mathcal{L}(\mathrm{X},\mathrm{Y})}$, $Dg_j(x_0) \neq \theta_{\mathrm{X}'}$, $j \in J_0$, and also that the operator $DH(x_0)$ is surjective.

By Lyusternik's theorem, if H is continuously differentiable in a neighborhood of x_0 and if $DH(x_0)$ is surjective then, the tangent cone of the set Q_1 at x_0 is the kernell (or nucleus) of the linear operator $DH(x_0)$:
$$K_{\mathrm{T}} = K_{\mathrm{T}}(Q_1, x_0) = \mathcal{N}[DH(x_0)] = \{h \in \mathrm{X} \mid DH(x_0)(h) = \theta_{\mathrm{Y}}\}.$$

The dual cone K_{T}^* is the orthogonal (or annihilator) of the subspace K_{T}. Since $DH(x_0)$ is continuous and its range $\mathcal{R}(DH(x_0)) = DH(x_0)(\mathrm{X}) = \mathrm{Y}$ is closed, we have (see [30]) $K_{\mathrm{T}}^* = (\mathcal{N}[DH(x_0)])^{\perp} = \mathcal{R}[DH(x_0)^*]$,where $DH(x_0)^*$ denotes the corresponding adjoint operator. Then, we conclude that
$$K_{\mathrm{T}}^* = \{f \in \mathrm{X}' \mid f = y^* \circ DH(x_0),\ y^* \in \mathrm{Y}'\ \}. \tag{4}$$

For Q_2 we introduce the following convex cone of feasible directions at x_0:
$$K_G = K_G(Q_2, x_0) = \{h \in \mathrm{X} \mid \exists \varepsilon_0 > 0 : G(x_0) + \varepsilon DG(x_0)(h) < \theta_{\mathrm{Z}}, \forall \varepsilon \in (0, \varepsilon_0)\},$$
and its "plus equality " analogous:
$$K_{G\leq} = \{h \in \mathrm{X} \mid \exists \varepsilon_0 > 0 : G(x_0) + \varepsilon DG(x_0)(h) \leq \theta_{\mathrm{Z}}, \forall \varepsilon \in (0, \varepsilon_0)\}.$$

It is not difficult to deduce the following properties:

1) $K_G \neq \emptyset \Rightarrow K_G^* = K_{G\leq}^*$,

2) P convex and $G(x_0) \leq \theta_{\mathrm{Z}} \Rightarrow K_{G_0} \subset K_G$ and $K_{G_0\leq} \subset K_{G\leq}$, where:

$$K_{G_0} = \{h \in \mathrm{X} \mid G(x_0) + DG(x_0)(h) < \theta_{\mathrm{Z}}\},$$

is a cone with apex at $G(x_0)$, and

$$K_{G_0\leq} = \{h \in \mathrm{X} \mid G(x_0) + DG(x_0)(h) \leq \theta_{\mathrm{Z}}\}$$

is the corresponding "plus equality " analogous.

With these properties and using the following natural extension of Farkas Lemma to cones with non zero apex (for a proof see [7]), we have a useful estimation of the dual cone of K_G.

Lemma 2: (Farkas Lemma) Let E_1, E_2 be normed spaces, $K_2 \subset E_2$ a convex cone with apex at x_2 such that the E_2-zero satisfies $\theta_2 \in cl(K_2)\backslash int(K_2)$. Define the convex cone:

$$K_1 = \{x_1 \in E_1 \mid A(x_1) \in K_2\},$$

where $A : E_1 \to E_2$ is a continuous linear operator, and suppose there exists a $\bar{x}_1 \in E_1$ such that:

$$A(\bar{x}_1) \in int(K_2).$$

Then, we have:

$$K_1^* = A^*(K_2^*) = \{f_1 \in E_1^* \mid f_1 = f_2 \circ A,\ f_2 \in K_2^*\},$$

where A^* denotes the adjoint operator corresponding to A.

Lemma 3: Suppose that $G(x_0) \leq \theta_{\mathrm{Z}}$ but $G(x_0) \not< \theta_{\mathrm{Z}}$, i.e.:

$$G(x_0) \in -\left(P \setminus int\ (P)\right), \tag{5}$$

and $K_{G_0} \neq \varnothing$, i.e. there exists $\overline{h} \in \mathrm{X}$, such that:

$$G(x_0) + DG(x_0)(\overline{h}) \in -\left(int(P)\right). \tag{6}$$

Then

$$K_G^* \subset \{f \in \mathrm{X}' \mid f = -z^* \circ DG(x_0),\ z^* \in \mathrm{Z}',\ z^* \geq \theta_{\mathrm{Z}'},\ z^*\left(G(x_0)\right) = 0\}. \tag{7}$$

Lemma 3 is the key for the calculation of the dual cone K_G^*, and (6) is a crucial hypothesis in order to use the Farkas' Lemma extension. It means that an awakening of the assumption $int(P) \neq \emptyset$ demands a finer generalization of Farkas' Lemma, which is not available so far.

On the other hand, it is well known that:

$$K_j = K_j(Q_{3j}, x_0) = \{h \in \mathrm{X} \mid Dg_j(x_0)(h) < 0\}\,, \quad j \in J_0$$

$$K_{\mathcal{X}} = K_{\mathcal{X}}(Q_4, x_0) = \{h \in \mathrm{X} \mid h = \lambda(x - x_0)\ , \lambda > 0\ ,\ x \in int\mathcal{X}\}\,,$$

are open convex cones of feasible directions at x_0, for the sets Q_{3j} and Q_4 respectively. Also,

$$K_0 = \{h \in \mathrm{X} \mid DF(x_0)(h) < 0\}$$

is a cone of decreasing directions of F at x_0 and in the non empty case, the dual cones have the form:

$$\begin{aligned} K_j^* &= \left\{f \in \mathrm{X}' \mid f = \mu_j Dg_j(x_0)\ , \mu_j \le 0\right\}, \quad j \in J_0, \\ K_{\mathcal{X}}^* &= \{f \in \mathrm{X}' \mid f(x) \ge f(x_0), \quad \forall x \in \mathcal{X}\}\,, \\ K_0^* &= \{f \in \mathrm{X}' \mid f = \lambda_0 DF(x_0)\ , \lambda_0 \le 0\}\,. \end{aligned}$$

The cone K_G has non empty interior because it contains the open set K_{G_0} which is nonempty. Then, classical arguments can be used to prove that the following intersection of cones is empty:

$$K_0 \cap int(K_G) \bigcap_{j \in J_0} K_j \cap K_T \cap K_{\mathcal{X}} = \emptyset.$$

By Dubovitskii-Milyutin's Lemma, there exist functionals $f_0 \in K_0^*$, $f_G \in (intK_G)^* = K_G^*$, $f_j \in K_j^*$, $j \in J_0$, $f_{\mathrm{T}} \in K_{\mathrm{T}}^*$, and $f_{\mathcal{X}} \in K_{\mathcal{X}}^*$ such that:

$$f_0 + f_G + \sum_{j \in J_0} f_j + f_{\mathrm{T}} + f_{\mathcal{X}} = 0. \tag{8}$$

Assuming $G(x_0) \in -(P \setminus int\ (P))$ and using the form of the dual cones and Lemma 3 we have:

$$\lambda_0 DF(x_0) + y^* \circ DH(x_0) + z^* \circ DG(x_0) + \sum_{j=1}^{n} \mu_j Dg_j(x_0) = f_{\mathcal{X}},$$

where $z^* \ge \theta_{\mathrm{Z}'}$, $z^*\,[G(x_0)] = 0$ and $f_{\mathcal{X}}$ is a support functional of $\mathcal{X}$ at x_0. Taking $\mu_j = 0$ for $j \notin J_0$ we obtain $\mu_j \ge 0$ $(j = \overline{1,n})$ and $\mu_j g_j(x_0) = 0$, $j = \overline{1,n}$. This is exactly the first thesis of the theorem.

The case when $G(x_0) < \theta_Z$ produces a trivial feasible cone $K_G = \mathrm{X}$, and its dual consists of the null functional $K^* = \{\theta_{\mathrm{X}'}\}$. Then, the result is the same but with a corresponding null multiplier $z^* = \theta_{\mathrm{Z}'}$.

Finally, if in addition to (2), the vector $\bar{h}$ satisfies $Dg_j(x_0)(\overline{h}) < 0$, $j \in J_0(x_0)$, $\overline{h} = \lambda(x - x_0)$, $\lambda > 0$, $x \in int\mathcal{X}$, and $DH(x_0)(\overline{h}) = \theta_{\mathrm{Y}}$ then, we would have $int(K_G) \bigcap\limits_{j\in J_0} K_j \cap K_T \cap K_{\mathcal{X}} \neq \emptyset$ and this implies $\lambda_0 > 0$. Therefore it can be taken $\lambda_0 = 1$. ■.

4 Optimal Control Problem with phase constraints

Let us consider the following optimal control problem:

$$\begin{array}{ll} \min & \int\limits_{t_0}^{t_1} f(x,u,t)\ dt \\ s.t. & \dot{x}(t) = h(x,u,t),\ a.e.\ t \in [t_0,t_1] \\ & x(t_0) = \overline{x_0}, \quad x(t_1) = \overline{x_1} \\ & g(x(t),u(t),t) \leq 0_p,\ a.e.\ t \in [t_0,t_1] \\ & q(x(t),t) \leq 0_q, \quad \forall t \in [t_0,t_1] \\ & x \in (\mathcal{C}^1_\infty([t_0,t_1],\Re^n), \|.\|_0) \\ & u \in (L_\infty([t_0,t_1],\Re^m), \|.\|_\infty) \\ & u \in \mathcal{U} = \{u \in L_\infty \mid u(t) \in U,\ a.e.\ t \in [t_0,t_1]\} \end{array} \tag{9}$$

where $\mathcal{C}^1_\infty$ is the set of absolutely continuous functions with derivative in L_∞. The functions $f : \Re^n \times \Re^m \times [t_0,t_1] \longrightarrow \Re$, $h : \Re^n \times \Re^m \times [t_0,t_1] \longrightarrow \Re^n$, $g : \Re^n \times \Re^m \times [t_0,t_1] \longrightarrow \Re^p$ and $q : \Re^n \times [t_0,t_1] \longrightarrow \Re^q$ are supposed to be continuous and continuously differentiable with respect to (x,u) and x, respectively. $U \subset \Re^m$ is a compact convex set with non empty interior and this implies that $\mathcal{U}$ is also a closed convex set with non empty interior in $L_\infty([t_0,t_1],\Re^m)$.

Let be $\mathrm{X} = \left(\mathcal{C}^1_\infty \times L_\infty, \|.\|_0 \times \|.\|_\infty\right)$, where $\|.\|_0$ and $\|.\|_\infty$ denote the uniform norm and the essential supremum norm, respectively, and define $\mathcal{X} = \mathcal{C}^1_\infty \times \mathcal{U}$. We introduce the following functionals and operators:

1. $F : \mathrm{X} \to \Re, \quad F(x,u) = \int\limits_{t_0}^{t_1} f(x,u,t)\ dt,$

2. $H : \mathrm{X} \longrightarrow \left(\mathcal{C}^1_\infty, \|.\|_0\right)$,

$$H(x,u)(t) = x(t) - \overline{x_0} - \int_{t_0}^{t} h(x(\tau),u(\tau),\tau)\ d\tau,\ t \in [t_0,t_1],$$

3. $E_{t_1} : \mathrm{X} \longrightarrow \Re^n, \quad E_{t_1}(x,u) = x(t_1) - \overline{x_1},$

4. $\overline{H} : \mathrm{X} \longrightarrow \mathcal{C}^1_\infty \times \Re^n, \quad \overline{H}(x,u) = (H(x,u), E_{t_1}(x,u))^{\mathrm{T}},$

5. $G : \mathrm{X} \longrightarrow (L_\infty([t_0,t_1], \Re^p), \|.\|_\infty), \quad G(x,u)(t) = g(x(t),u(t),t), \quad t \in [t_0,t_1],$ where we consider the set L_∞ as a normed space, ordered by the following convex and proper cone:

$$\mathrm{P} = \{p \in L_\infty \mid p(t) \geq 0_p, \ a.e. \ t \in [t_0,t_1]\}. \tag{10}$$

6. $Q : \mathcal{C}^1_\infty \longrightarrow (\mathcal{C}([t_0,t_1], \Re^q), \|.\|_0), \quad Q(x)(t) = q(x(t),t), \ t \in [t_0,t_1],$

   where we also consider the set of continuous $q-$vector functions, $\mathcal{C} = \mathcal{C}([t_0,t_1], \Re^q)$, as a normed space, ordered by the following convex and proper cone:

$$\mathrm{P}_q = \{q \in \mathcal{C}([t_0,t_1], \Re^q) \mid q(t) \geq 0_q, \ \forall \ t \in [t_0,t_1]\}. \tag{11}$$

7. $\overline{G} : \mathrm{X} \longrightarrow L_\infty \times \mathcal{C}, \quad \overline{G}(x,u) = (G(x,u), Q(x))^{\mathrm{T}}.$

Then, the problem (9) can be written in the following form:

$$\begin{array}{ll} \min & F(x,u) \\ s.a. & \overline{H}(x,u) = \theta_1, \\ & \overline{G}(x,u) \leq \theta_2, \\ & (x,u) \in \mathcal{X}, \end{array} \tag{12}$$

where $\theta_1 = (\theta_n, 0_n)^{\mathrm{T}}$ and $\theta_2 = (\theta_p, \theta_q)^{\mathrm{T}}$ denote the zeros of the spaces $\mathcal{C}^1_\infty([t_0,t_1], \Re^n) \times \Re^n$ and $L_\infty([t_0,t_1], \Re^p) \times \mathcal{C}([t_0,t_1], \Re^q)$ respectively.

Furthermore, the operators $F, \overline{H}$ and $\overline{G}$ are continuously Frechet differentiable, and the following formulae hold:

$$D_x F(x,u)(r) = \int_{t_0}^{t_1} f_x(x,u,t) r(t) \ dt, \tag{13}$$

$$D_u F(x,u)(v) = \int_{t_0}^{t_1} f_u(x,u,t) v(t) \ dt, \tag{14}$$

$D\overline{H}(x,u)(r,v) = (DH(x,u)(r,v), E_{t_1}(x,u))^{\mathrm{T}}$, where:

$$DH(x,u) = (D_x H(x,u), D_u H(x,u)),$$

$$\begin{aligned} D_x H(x,u)(r)(t) &= r(t) - \int_{t_0}^{t} h_x(x,u,\tau) r(\tau)\, d\tau, && (15) \\ D_u H(x,u)(v)(t) &= -\int_{t_0}^{t} h_u(x,u,\tau) v(\tau)\, d\tau. && (16) \end{aligned}$$

Similarly:

$$D\overline{G}(x,u) = \begin{pmatrix} D_x G & D_u G \\ D_x Q & D_u Q \end{pmatrix} (x,u),$$

$$\begin{aligned} D_x\overline{G}(x,u)(r)(t) &= (g_x(x(t),u(t),t)r(t), q_x(x(t),t)r(t))^{\mathrm{T}}, && (17) \\ D_u\overline{G}(x,u)(v)(t) &= (g_u(x(t),u(t),t)v(t), 0)^{\mathrm{T}}. && (18) \end{aligned}$$

Next, we shall use the following:

Lemma 4: If the variational system:

$$\dot{r}(t) = h_x(\overline{x},\overline{u},t)r(t) + h_u(\overline{x},\overline{u},t)v(t),\ a.e.\ t \in [t_0,t_1] \tag{19}$$

is full controllable, then the operator $D\overline{H}(\overline{x},\overline{u})$ is surjective.

A proof can be found in [13] or [14].

Definition 5: A feasible solution $(\bar{x},\bar{u})$ of the problem (9) is called *regular* if the variational system (19) is full controllable and

$$\begin{aligned} rank\,(g_u(\bar{x}(t),\bar{u}(t),t)) &= p\ ,\ a.e.\ t \in [t_0,t_1]\,. \\ rank\,(q_x(\bar{x}(t),t)) &= q\ ,\ \forall t \in [t_0,t_1]\,. \end{aligned}$$

Theorem 6 (Pontryagin's Maximum Principle): Let us consider the optimal control problem (9), with the given above assumptions. Let (x_0,u_0) be a regular local minima of the problem and suppose that the following condition holds:

$$\exists \mathcal{V} \mid \mathcal{V} \subset g_u(x_0(t),u_0(t),t)\,(U - u_0(t))\,, \quad a.e.\ t \in [t_0,t_1]\,, \tag{20}$$

where $\mathcal{V} = \mathcal{V}(0_p)$ is a neighborhood of $0_p \in \Re^p$.

Then, there exist multipliers $\lambda_0 \geq 0, \lambda^{\mathrm{T}} = (\lambda_1, ..., \lambda_n)$, not all zeros, a function $\rho_0 \in L_\infty [t_0, t_1]$, $\rho_0(t) \geq 0$, *a.e.* $t \in [t_0, t_1]$, which can be non zero only in the set:

$$R_0 = \{t \in [t_0, t_1] \mid g(x_0(t), u_0(t), t) = 0_p\},$$

or, more precisely, which satisfies:

$$\rho_0(t)\ g(x_0(t), u_0(t), t) = 0_p, \quad a.e.\ t \in [t_0, t_1],$$

a complete and positive Borel measure η with support in the set:

$$R_1 = \{t \in [t_0, t_1] \mid q(x_0(t), t) = 0_q\},$$

and a continuous function $\psi_0(t) : [t_0, t_1] \to \Re^n$, such that, ψ_0 is the solution of the integral equation:

$$-\psi^{\mathrm{T}}(t) = -\int_t^{t_1} \mathcal{H}_x^{\mathrm{T}}(x_0(t), \psi(t), \rho_0(t), u_0(t), t)\ d\tau + \int_t^{t_1} q_x(x_0, \tau)\ d\eta + \lambda^{\mathrm{T}}, \tag{21}$$

and moreover, the (local) maximum condition:

$$[-\mathcal{H}_u(x_0(t), \psi_0(t), \rho_0(t), u_0(t), t)]\,(u - u_0(t)) \geq 0, \quad \forall u \in U, \quad a.e.\ t \in [t_0, t_1], \tag{22}$$

holds, where $\mathcal{H}(x, \psi, \rho, u, t)$ is the Hamiltonian, defined by:

$$\mathcal{H}(x, \psi, \rho, u, t) = \psi^{\mathrm{T}} h(x, u, t) + \rho g(x, u, t) - \lambda_0 f(x, u, t).$$

Proof. The problem (9) is equivalent to (12) and we shall show that we can apply KKT theorem 1. In fact, we have differentiability of the functional and operators and $D\overline{H}(x_0, u_0)(\mathrm{X})$ is closed because, by regularity, $D\overline{H}(x_0, u_0)$ is surjective (see Lemma 4). In addition, $\mathcal{X}$ is a convex set with non empty interior since $\mathcal{U}$ has the same properties. We still have to check the third assumption about inequalities, and to do that we shall prove the following:

Lemma 7: If (x_0, u_0) is regular, then there is a $(\overline{x}, \overline{u}) \in \mathrm{X}$, such that:

$$\overline{G}(x_0, u_0) + D\overline{G}(x_0, u_0)(\overline{x}, \overline{u}) < \theta_2 \tag{23}$$

Proof. The expression (23) is equivalent to:

$$\overline{G}(x_0, u_0)(t) + D\overline{G}(x_0, u_0)(\overline{x}, \overline{u})(t) < \theta_2(t) = (0_p, 0_q),$$

or

$$\begin{cases} G(x_0, u_0)(t) + DG(x_0, u_0)(\overline{x}, \overline{u})(t) < 0_p,\ a.e.\ t \in [t_0, t_1], \\ Q(x_0)(t) + DQ(x_0)(\overline{x})(t) < 0_q,\ \forall\ t \in [t_0, t_1], \end{cases} \tag{24}$$

i.e.,

$$\begin{cases} g(x_0(t),u_0(t),t)+g_x(x_0(t),u_0(t),t)\overline{x}(t)+g_u(x_0(t),u_0(t),t)\overline{u}(t)<0_p \ ,a.e.\ t\in[t_0,t_1]\,, \\ q(x_0(t),t)+q_x(x_0(t),t)\overline{x}(t)<0_q \ , \ \forall\, t\in[t_0,t_1]\,. \end{cases}$$

Take $\tilde{x}(t) = 0_n$ for all $t \in [t_0,t_1]$. If $t \notin R_0$, the first inequality holds with $\overline{u}(t) = 0_m$, because $g(x_0(t),u_0(t),t) < 0_p$. If $t \in R_0$ then, choose $\widetilde{u}(t) = g_u^{\mathrm{T}}(x_0(t),u_0(t),t)v_1(t)$ for all $t \in R_0$, where $v_1(t)$ is the solution of the system:

$$\left(g_u(x_0(t),u_0(t),t)g_u^{\mathrm{T}}(x_0(t),u_0(t),t)\right)v_1(t) = -\varepsilon_p,$$

$\varepsilon_p = (\varepsilon,\varepsilon,\cdots,\varepsilon) \in \Re^p$ and $\varepsilon > 0$. This system has a solution because the matrix $\left(g_u(x_0(t),u_0(t),t)g_u^{\mathrm{T}}(x_0(t),u_0(t),t)\right)$ is invertible as a consequence of regularity. Now, defining the measurable function $\overline{u}(t) = 1_{R_0}(t)\widetilde{u}(t)$ for $t \in [t_0,t_1]$, we have a pair $(\tilde{x},\overline{u}) \in \mathrm{X}$ such that, the first inequality of (24) holds.

The operator $G(x_0,u_0)+DG(x_0,u_0)(x,u)$ is continuous with respect to (x,u), and its image at $(\tilde{x},\bar{u}) = (\theta_n,\bar{u})$ belongs to $int(-\mathrm{P})$. Then, there is an open neighborhood $B = B_0 \times B_\infty$, defined by:

$$B_0 = B(\tilde{x},\varepsilon) = \left\{x \in \mathcal{C}^1_\infty \mid \|x-\tilde{x}\|_0 < \varepsilon\right\},$$

and

$$B_\infty = B(\overline{u},\varepsilon) = \left\{u \in L_\infty \mid \|u-\overline{u}\|_\infty < \varepsilon\right\},$$

such that:

$$G(x_0,u_0)+DG(x_0,u_0)(B) \in int(-\mathrm{P}), \tag{25}$$

We shall prove that:

$$\{Q(x_0)+DQ(x_0)(B_0)\} \cap int(-\mathrm{P}_q) \neq \emptyset. \tag{26}$$

Let be $x_n(t) = q_x^{\mathrm{T}}(x_0(t),t)x_n'(t)$ and $\varepsilon_q^n = (\underbrace{\frac{1}{n},\frac{1}{n},...,\frac{1}{n}}_{\text{q times}})$ where $x_n'(t)$ is the solution of the system:

$$q_x(x_0(t),t)q_x^{\mathrm{T}}(x_0(t),t)x_n'(t) = -\varepsilon_q^n.$$

Since $rank\,(q_x(x_0(t),t)) = q,\ \forall t \in [t_0,t_1]$, this solution exists for all $t \in [t_0,t_1]$. Furthermore, $x_n \underset{n}{\longrightarrow} \tilde{x} = \theta_n$, in the norm $\|.\|_0$, because:

$$x_n(t) = -q_x^{\mathrm{T}}(x_0(t),t)\left(q_x(x_0(t),t)q_x^{\mathrm{T}}(x_0(t),t)\right)^{-1}\varepsilon_q^n,$$

and $q_x^{\mathrm{T}}(x_0(t),t)\left(q_x(x_0(t),t)q_x^{\mathrm{T}}(x_0(t),t)\right)^{-1}$ is a continuous map respect to t and hence, is bounded for all $t \in [t_0,t_1]$. We conclude that, for n big enough, the pair $(\bar{x},\bar{u}) = (x_n,\bar{u}) \in B$ satisfies (25) and:

$$q(x_0(t),t) + q_x(x_0(t),t)\bar{x}(t) = q(x_0(t),t) - \varepsilon_q^n < 0_q, \ \ \forall t \in [t_0,t_1]\,. \ \blacksquare$$

Hence, the assumptions of the KKT theorem 1 are fulfilled, and for its application we introduce the functionals:

$$h_j(x) = x_j(t_1) - \overline{x_1}_j,$$

$h_j : \mathcal{C}^1_\infty \longrightarrow \Re, \forall j = \overline{1,n}$, where $x_j(t_1), \overline{x_1}_j$ are j-th coordinates of $x(t_1)$ and $\overline{x_1}$, respectively. Then, we can write:

$$E_{t_1}(x,u) = (h_1(x), h_2(x), \cdots, h_n(x))\,.$$

In addition, we shall use that, for $(x_0,u_0) \in \mathrm{X}$, $D_x H(x_0,u_0)$ is an homeomorphism and its inverse has the form:

$$D_x H^{-1}(x_0,u_0)(y)(t) = y(t) + e^{A(t)} \int_{t_0}^{t} e^{-A(\tau)} h_x(x_0(\tau),u_0(\tau),\tau) y(\tau)\ d\tau, \tag{27}$$

where:

$$A(t) = \int_{t_0}^{t} h_x(x_0(\tau),u_0(\tau),\tau)\ d\tau. \tag{28}$$

(see [13] or [14]).

Applying theorem 1, there exist $\lambda_0 \in \Re$, $\lambda_j \in \Re$, $j = \overline{1,n}$, $y^* \in (L_\infty)'$, $z^* \in \left(\mathcal{C}^1_\infty\right)'$, $s^* \in (\mathcal{C})'$ and a support functional f of $\mathcal{U}$ in u_0 such that:

$$\lambda_0 D_u F(x_0,u_0) + z^* \circ D_u H(x_0,u_0) + y^* \circ D_u G(x_0,u_0) = f, \tag{29}$$

$$\lambda_0 D_x F(x_0,u_0) + z^* \circ D_x H(x_0,u_0) + y^* \circ D_x G(x_0,u_0) + s^* \circ D_x Q(x_0) + \sum_{j=1}^{n} \lambda_j D h_j(x_0) = \theta_n^* \tag{30}$$

From equation (30), applying $D_x H^{-1}(x_0,u_0)$, we have for any $y \in \mathcal{C}^1_\infty$:

$$\begin{aligned} z^*(y) = \ & -\lambda_0 D_x F(x_0,u_0) \circ D_x H^{-1}(x_0,u_0)(y) - y^* \circ DG(x_0,u_0) \circ D_x H^{-1}(x_0,u_0)(y) \\ & - s^* \circ D_x Q(x_0) \circ D_x H^{-1}(x_0,u_0)(y) - \sum_{j=1}^{n} \lambda_j D h_j(x_0) \circ D_x H^{-1}(x_0,u_0)(y). \end{aligned} \tag{31}$$

But $Dh_j(x_0)(r) = r_j(t_1)$, hence:

$$\sum_{j=1}^{n} \lambda_j Dh_j(x_0)\left(D_x H^{-1}(x_0,u_0)(y)\right) = \lambda^{\mathrm{T}} y(t_1) + \lambda^{\mathrm{T}} e^{A(t_1)} \int_{t_0}^{t_1} e^{-A(\tau)} h_x(x_0,u_0,\tau) y(\tau)\, d\tau, \tag{32}$$

where $\lambda^{\mathrm{T}} = (\lambda_1, ..., \lambda_n)$ and $A(t)$ is defined in (28). Then, substituting (13), (17), (27) and (32) in (31) we obtain:

$$\begin{aligned} z^*(y) = & -\int_{t_0}^{t_1} \phi^{\mathrm{T}}(\tau) y(\tau)\, d\tau - \lambda^{\mathrm{T}} y(t_1) - \lambda^{\mathrm{T}} e^{A(t_1)} \int_{t_0}^{t_1} e^{-A(\tau)} h_x(x_0,u_0,\tau) y(\tau)\, d\tau - \\ & - y^* \left[g_x(x_0,u_0,t) y(t)\right] - y^* \left[g_x(x_0,u_0,t) e^{A(t)} \int_{t_0}^{t} e^{-A(\tau)} h_x(x_0,u_0,\tau) y(\tau)\, d\tau\right] - \\ & - s^* \left[q_x(x_0,t) y(t)\right] - s^* \left[q_x(x_0(t),t) e^{A(t)} \int_{t_0}^{t} e^{-A(\tau)} h_x(x_0,u_0,\tau) y(\tau)\, d\tau\right], \end{aligned} \tag{33}$$

with $\phi^{\mathrm{T}}(\tau) = \lambda_0 \left[f_x(x_0,u_0,\tau) + \left(\int_{\tau}^{t_1} f_x(x_0,u_0,t) e^{A(t)}\, dt\right) e^{-A(\tau)} h_x(x_0,u_0,\tau)\right]$.

Using (33) in the equation (30), evaluated at $y \in \mathcal{C}^1_\infty$, and with some transformations (see [8] or [9]) we obtain:

$$\begin{aligned} & \int_{t_0}^{t_1} \left[\lambda_0 f_x(x_0,u_0,t) - \phi^{\mathrm{T}}(t) + \lambda^{\mathrm{T}} h_x(x_0,u_0,t) - \lambda^{\mathrm{T}} e^{A(t_1)} e^{-A(t)} h_x(x_0,u_0,t)\right] y(t)\, dt \\ & + \int_{t_0}^{t_1} \left[\int_{t}^{t_1} \left(\phi^{\mathrm{T}}(\tau) + \lambda^{\mathrm{T}} e^{A(t_1)} e^{-A(\tau)} h_x(x_0,u_0,\tau)\right) d\tau\right] h_x(x_0,u_0,t) y(t)\, dt \\ & + y^* \left[g_x(x_0,u_0,t) \int_{t_0}^{t} h_x(x_0,u_0,\tau) y(\tau)\, d\tau\right] \\ & - y^* \left[g_x(x_0,u_0,t) e^{A(t)} \int_{t_0}^{t} e^{-A(\tau)} h_x(x_0,u_0,\tau) y(\tau)\, d\tau\right] \\ & + y^* \left[g_x(x_0,u_0,t) e^{A(t)} \int_{t_0}^{t} e^{-A(\tau)} h_x(x_0,u_0,\tau) \int_{t_0}^{\tau} h_x(x_0,u_0,s) y(s)\, ds\, d\tau\right] \\ & + s^* \left[q_x(x_0,t) \int_{t_0}^{t} h_x(x_0,u_0,\tau) y(\tau)\, d\tau\right] - s^* \left[q_x(x_0,t) e^{A(t)} \int_{t_0}^{t} e^{-A(\tau)} h_x(x_0,u_0,\tau) y(\tau)\, d\tau\right] \\ & + s^* \left[q_x(x_0,t) e^{A(t)} \int_{t_0}^{t} e^{-A(\tau)} h_x(x_0,u_0,\tau) \int_{t_0}^{\tau} h_x(x_0,u_0,s) y(s)\, ds\, d\tau\right] = 0. \end{aligned} \tag{34}$$

From Theorem 1 , $y^* \geq 0$ and $s^* \geq 0$, hence, using Riesz's representation theorem, there exist positive and complete Borel measures $\mu(t)$ and $\eta(t)$, such that:

$$y^*\left[y(t)\right] = \int_{t_0}^{t_1} y(t) d\mu(t),$$

$$s^*\left[y(t)\right] \;\; = \int_{t_0}^{t_1} y(t) d\eta(t),$$

and using Fubini's Theorem, we obtain:

$$y^*\left[g_x(x_0,u_0,t)\int_{t_0}^{t} h_x(x_0,u_0,\tau)y(\tau)\ d\tau\right] = \int_{t_0}^{t_1}\left(\int_{t}^{t_1} g_x(x_0,u_0,\tau)\ d\mu(\tau)\right) h_x(x_0,u_0,t)y(t)\ dt,$$

$$s^*\left[q_x(x_0,t)\int_{t_0}^{t} h_x(x_0,u_0,\tau)y(\tau)\ d\tau\right] = \int_{t_0}^{t_1}\left(\int_{t}^{t_1} q_x(x_0,\tau)\ d\eta(\tau)\right) h_x(x_0,u_0,t)y(t)\ dt. \quad (35)$$

In the same way, we arrive to analogous relations for:

$$y^*\left[g_x(x_0,u_0,t)e^{A(t)}\int_{t_0}^{t} e^{-A(\tau)}h_x(x_0,u_0,\tau)y(\tau)\ d\tau\right],$$
$$y^*\left[g_x(x_0,u_0,t)e^{A(t)}\int_{t_0}^{t} e^{-A(\tau)}h_x(x_0,u_0,\tau)\int_{t_0}^{\tau} h_x(x_0,u_0,s)y(s)\ ds\right],$$
$$s^*\left[q_x(x_0,t)e^{A(t)}\int_{t_0}^{t} e^{-A(\tau)}h_x(x_0,u_0,\tau)y(\tau)\ d\tau\right],$$
$$\text{and } s^*\left[q_x(x_0,t)e^{A(t)}\int_{t_0}^{t} e^{-A(\tau)}h_x(x_0,u_0,\tau)\int_{t_0}^{\tau} h_x(x_0,u_0,s)y(s)\ ds\right].$$

Substituting in (34) we obtain:

$$\begin{aligned}
&\int_{t_0}^{t_1}\left[\lambda_0 f_x(x_0,u_0,t)-\phi^{\mathrm{T}}(t)+\lambda^{\mathrm{T}}h_x(x_0,u_0,t)-\lambda^{\mathrm{T}}e^{A(t_1)}e^{-A(t)}h_x(x_0,u_0,t)\right]y(t)\ dt-\\
&\qquad -\int_{t_0}^{t_1}\left[\left(\int_t^{t_1} g_x(x_0,u_{0,}\tau)e^{A(\tau)}\ d\mu(\tau)\right)e^{-A(t)}h_x(x_0,u_0,t)\right]y(t)\ dt-\\
&\qquad -\int_{t_0}^{t_1}\left[\left(\int_t^{t_1} q_x(x_0,\tau)e^{A(\tau)}\ d\eta(\tau)\right)e^{-A(t)}h_x(x_0,u_0,t)\right]y(t)\ dt+\\
&\qquad +\int_{t_0}^{t_1}\left[\int_t^{t_1}\left(\phi^{\mathrm{T}}(\tau)+\lambda^{\mathrm{T}}e^{A(t_1)}e^{-A(\tau)}h_x(x_0,u_0,\tau)\right)d\tau\right]h_x(x_0,u_0,t)y(t)\ dt+\\
&\qquad +\int_{t_0}^{t_1}\left[\int_t^{t_1} g_x(x_0,u_{0,}\tau)\ d\mu(\tau)+\int_t^{t_1} q_x(x_0,\tau)\ d\eta(\tau)\right]h_x(x_0,u_0,t)y(t)\ dt+\\
&\quad +\int_{t_0}^{t_1}\left[\int_t^{t_1} g_x(x_0,u_{0,}\tau)e^{A(\tau)}\left(\int_t^{\tau} e^{-A(s)}h_x(x_0,u_0,s)\ ds\right)\ d\mu(\tau)\right]h_x(x_0,u_0,t)y(t)\ dt+\\
&\quad +\int_{t_0}^{t_1}\left[\int_t^{t_1} q_x(x_0,\tau)e^{A(\tau)}\left(\int_t^{\tau} e^{-A(s)}h_x(x_0,u_0,s)\ ds\right)\ d\eta(\tau)\right]h_x(x_0,u_0,t)y(t)\ dt=0,
\end{aligned}$$

Defining:

$$\begin{aligned}
\psi^{\mathrm{T}}(t)= \quad & -\int_t^{t_1}\left(\phi^{\mathrm{T}}(\tau)+\lambda^{\mathrm{T}}e^{A(t_1)}e^{-A(\tau)}h_x(x_0,u_0,\tau)\right)d\tau-\int_t^{t_1} g_x(x_0,u_0,\tau)\ d\mu(\tau)\\
&-\int_t^{t_1} q_x(x_0,\tau)\ d\eta(\tau)-\int_t^{t_1} g_x(x_0,u_0,\tau)e^{A(\tau)}\left(\int_t^{\tau} e^{-A(s)}h_x(x_0,u_0,s)\ ds\right)\ d\mu(\tau)\\
&-\int_t^{t_1} q_x(x_0,\tau)e^{A(\tau)}\left(\int_t^{\tau} e^{-A(s)}h_x(x_0,u_0,s)\ ds\right)\ d\eta(\tau)-\lambda^{\mathrm{T}},
\end{aligned} \tag{36}$$

we arrive to the equation:

$$\begin{aligned}
-\psi^{\mathrm{T}}(t)= \quad & \lambda^{\mathrm{T}}+\int_t^{t_1}\left[\lambda_0 f_x(x_0,u_0,\tau)-\psi^{\mathrm{T}}(\tau)h_x(x_0,u_0,\tau)\right]\ d\tau\\
&+\int_t^{t_1} g_x(x_0,u_0,\tau)\ d\mu(\tau)+\int_t^{t_1} q_x(x_0,\tau)\ d\eta(\tau).
\end{aligned} \tag{37}$$

With a similar analysis we deduce that, for all $v\in L_\infty$:

$$\int_{t_0}^{t_1}\left[\lambda_0 f_u(x_0,u_0,t)-\psi^{\mathrm{T}}(t)h_u(x_0,u_0,t)\right]v(t)\ dt+y^*\left[g_u(x_0,u_0,t)v(t)\right]=f(v). \tag{38}$$

Now we shall prove that, if the condition (20) holds, the functional y^* can be represented, in integral form, by a function of L_∞. For this end, we recall first the following Lemma, a proof of which can be found in [16].

Lemma 8 (Measurable Selection): Let be the set $D \subset \Re^{m+1}$ and define:

$$\begin{aligned} D^t &= \{v \in \Re^m \mid (t,v) \in D\}\,, \\ \Delta &= \left\{t \mid D^t \neq \emptyset\right\}. \end{aligned}$$

If the set D is σ-compact (finitely denumerable union of compact sets) then, there exist a measurable function $v(t)$ such that $(t, v(t)) \in D$, for almost all $t \in \Delta$.

Second, we need a representation of some special simple functions:

Lemma 9: Let's suppose that (x_0, u_0) satisfies the condition (20) then, for any function of the form $1_E(t)a$ with its image in $\mathcal{V}(\theta_p)$ (where $a \in \Re^p$and $E \subset [t_0, t_1]$ is a measurable set), there exist $\overline{v} \in \mathcal{U}$ such that:

$$g_u(x_0, u_0, t)\,(\overline{v}(t) - u_0(t)) = 1_E(t)a, \quad a.e.\ t \in [t_0, t_1]\,.$$

Proof. Because $Im\,[1_E(t)a] \subset \mathcal{V}(\theta_p)$ and the regularity condition (20), for all $t \in E$, there exist $v \in U$ such that:

$$g_u(x_0, u_0, t)\,(v - u_0(t)) = 1_E(t)a.$$

But $1_E(t)a$ is a measurable function in $[t_0, t_1]$, then by Luzin's theorem, there exist a sequence of compact sets $\Delta_1 \subset \Delta_2 \subset ...$ such that:

$$[t_0, t_1] \setminus (\Delta_1 \cup \Delta_2 \cup ...)$$

has null measure and the restriction of $1_E(t)a$ to each Δ_i is a continuous function. Let's define:

$$\begin{aligned} D_i &= \{(t,v) \mid t \in \Delta_i\ ,\ v \in U\ ,\ g_u(x_0, u_0, t)\,(v - u_0(t)) = 1_E(t)a\}\,, \\ D &= D_1 \cup D_2 \cup ... \quad , \quad \Delta = \Delta_1 \cup \Delta_2 \cup ... \end{aligned}$$

Let's also define:

$$\begin{aligned} & D^t = \{v \in \Re^m \mid (t,v) \in D\} \\ \Rightarrow\quad & D^t = \{v \in U \mid g_u(x_0, u_0, t)\,(v - u_0(t)) = 1_E(t)a\}\,,\ t \in \Delta. \end{aligned}$$

It's clear that $\Delta = \{t \mid D^t \neq \emptyset\}$, because if $t \in \Delta$ then there exist $i \in \aleph$ such that $t \in \Delta_i$ and by the regularity condition (20) we have $D_i \neq \emptyset$ and hence $D^t \neq \emptyset$.

Conversely, if t is such that $D^t \neq \emptyset$ then $t \in \Delta$.

Now, for σ-compactness we have only to prove that D_i is compact for all $i \in \aleph$. In fact, $D_i \subset \Delta_i \times U$ where Δ_i, U, are compacts and D_i is a closed set, because for any sequence $(t_n, v_n) \in D_i$ such that $(t_n, v_n) \longrightarrow (\bar{t}, \overline{v})$ we have:

$$g_u(x_0(t_n), u_0(t_n), t_n)\,(v - u_0(t_n)) = 1_E(t_n)a.$$

Since the function $1_E(t)a$ is continuous in Δ_i then, the function $(t, v) \rightarrow g_u(x_0, u_0, t)\,(v - u_0(t))$ is also continuous with respect to t in Δ_i and also with respect to v, as a dot product. Hence:

$$\lim_n g_u(x_0(t_n), u_0(t_n), t_n)\,(v - u_0(t_n)) = g_u(x_0(\bar{t}), u_0(\bar{t}), \bar{t})\left(v - u_0(\bar{t})\right).$$

But

$$\lim_n g_u(x_0(t_n), u_0(t_n), t_n)\,(v - u_0(t_n)) = \lim_n 1_E(t_n)a = 1_E(\bar{t})a$$

and then $(\bar{t}, \overline{v}) \in D_i$, i.e., D is σ-compact. Applying Lemma 8, there exists a measurable function $\overline{v}(t)$ such that $(t, \overline{v}(t)) \in D$ for almost all $t \in \Delta$. Hence,

$$g_u(x_0, u_0, t)\,(\overline{v}(t) - u_0(t)) = 1_E(t)a, \quad a.e.\ t \in [t_0, t_1].$$

Finally, since U is bounded then, $\overline{v} \in L_\infty$ and this means that $\overline{v} \in \mathcal{U}$. ■

Lemma 10: Let (x_0, u_0) satisfy the condition (20), then y^* can be identified with a function $\rho_0 \in L_\infty([t_0, t_1], \Re^p)$.

Proof. The proof follows the same idea of Colonius [11]. Let us consider the subspace S of simple functions in $L_\infty([t_0, t_1], \Re^p)$. S is dense in $L_\infty([t_0, t_1], \Re^p)$. We shall prove that $y^* \mid S$ is continuous with respect to the norm in $L_1([t_0, t_1], \Re^p)$ on S. Then $y^* \mid S$ can be extended to a continuous linear functional $\overline{\rho_0}$ on $L_1([t_0, t_1], \Re^p)$ which by duality of L_1 and L_∞ can be identified with an element ρ_0 of $L_\infty([t_0, t_1], \Re^p)$. Then y^* and the functional $\overline{\rho_0}$,defined by ρ_0, coincide on S and hence on $L_\infty([t_0, t_1], \Re^p)$.

The general element $s \in S$ can be taken in the form:

$$s(t) = \sum_{i=1}^{l} \sum_{k=1}^{p} s_{ik} 1_{E_i}(t) a_k,\ t \in [t_0, t_1],$$

where $\{a_k\}_{k=1}^p$ is a basis of vectors in $\Re^p$ such that, $a_k \in \mathcal{V}(\theta_p)$ for all k ; $\{E_i\}_{i=1}^l$ is a measurable decomposition of the interval $[t_0, t_1]$ and $s_{ik} \in \Re$, for all i, k. Then,

$$\begin{aligned} y^*(s) &= y^*\left(\sum_{i=1}^{l}\sum_{k=1}^{p} s_{ik} 1_{E_i}(t) a_k\right) \\ &= \sum_{i=1}^{l}\sum_{k=1}^{p} (s_{ik}^+ - s_{ik}^-)\ y^*\left(1_{E_i}(t)a_k\right) \\ &= \sum_{i=1}^{l}\sum_{k=1}^{p} s_{ik}^+\ y^*\left(1_{E_i}(t)a_k\right) + \sum_{i=1}^{l}\sum_{k=1}^{p} s_{ik}^-\ y^*\left(-1_{E_i}(t)a_k\right) \end{aligned}$$

with s_{ik}^+ and s_{ik}^-, non negative real numbers. Applying Lemma 9, there exist v_{ik}^+ , $v_{ik}^- \in \mathcal{U}$ such that:

$$g_u(x_0, u_0, t)\left(v_{ik}^{\pm}(t) - u_0(t)\right) = \pm 1_{E_i}(t) a_k \quad \forall i, k$$

$$\begin{aligned} \Rightarrow y^*(s) = &\sum_{i=1}^{l}\sum_{k=1}^{p} s_{ik}^+\ y^*\left[g_u(x_0, u_0, t)\left(v_{ik}^+(t) - u_0(t)\right)\right] \\ &+ \sum_{i=1}^{l}\sum_{k=1}^{p} s_{ik}^-\ y^*\left[g_u(x_0, u_0, t)\left(v_{ik}^-(t) - u_0(t)\right)\right], \end{aligned}$$

From the expression (38), using the facts that f is a support functional of $\mathcal{U}$ at u_0 and that $\left(v_{ik}^{\pm}(t) - u_0(t)\right) = 0$ if $t \notin E_i$, we have:

$$\begin{aligned} &y^*\left[g_u(x_0, u_0, t)\left(v_{ik}^{\pm}(t) - u_0(t)\right)\right] \geq \\ &\geq -\int_{E_i}\left[\lambda_0 f_u(x_0, u_0, t) - \psi^{\mathrm{T}}(t) h_u(x_0, u_0, t)\right]\left(v_{ik}^{\pm}(t) - u_0(t)\right)\ dt. \end{aligned}$$

Hence,

$$\begin{aligned} y^*(s) \quad &\geq -\sum_{i=1}^{l}\sum_{k=1}^{p} s_{ik}^+ \int_{E_i}\left[\lambda_0 f_u(x_0, u_0, t) - \psi^{\mathrm{T}}(t) h_u(x_0, u_0, t)\right]\left(v_{ik}^+(t) - u_0(t)\right)\ dt \\ &\quad -\sum_{i=1}^{l}\sum_{k=1}^{p} s_{ik}^- \int_{E_i}\left[\lambda_0 f_u(x_0, u_0, t) - \psi^{\mathrm{T}}(t) h_u(x_0, u_0, t)\right]\left(v_{ik}^-(t) - u_0(t)\right)\ dt \end{aligned}$$

$$\begin{aligned} \Rightarrow y^*(s) \quad &\geq -\sum_{i=1}^{l}\sum_{k=1}^{p} s_{ik}^+ \sup_{t\in E_i}\left|\lambda_0 f_u(x_0, u_0, t) - \psi^{\mathrm{T}}(t) h_u(x_0, u_0, t)\right|\left|v_{ik}^+(t) - u_0(t)\right|\ \nu(E_i) \\ &\quad -\sum_{i=1}^{l}\sum_{k=1}^{p} s_{ik}^- \sup_{t\in E_i}\left|\lambda_0 f_u(x_0, u_0, t) - \psi^{\mathrm{T}}(t) h_u(x_0, u_0, t)\right|\left|v_{ik}^-(t) - u_0(t)\right|\ \nu(E_i) \end{aligned}$$

where ν denotes the Lebesgue measure in $[t_0, t_1]$.

But:

$$\begin{aligned} &\sup_{t\in E_i}\left|\lambda_0 f_u(x_0, u_0, t) - \psi^{\mathrm{T}}(t) h_u(x_0, u_0, t)\right|\left|v_{ik}^{\pm}(t) - u_0(t)\right| \leq \\ &\sup_{t\in[t_0,t_1]}\left|\lambda_0 f_u(x_0, u_0, t) - \psi^{\mathrm{T}}(t) h_u(x_0, u_0, t)\right| \sup_{t\in[t_0,t_1]}\left|v_{ik}^{\pm}(t) - u_0(t)\right| \leq \\ &\sup_{t\in[t_0,t_1]}\left|\lambda_0 f_u(x_0, u_0, t) - \psi^{\mathrm{T}}(t) h_u(x_0, u_0, t)\right| \sup_{v\in U}\sup_{t\in[t_0,t_1]}\left|v - u_0(t)\right|, \end{aligned}$$

and $\left|\lambda_0 f_u(x_0,u_0,t) - \psi^{\mathrm{T}}(t) h_u(x_0,u_0,t)\right|$ is continuous respect to t and bounded. Furthermore, U is compact and then $\sup\limits_{v\in U}\ \sup\limits_{t\in[t_0,t_1]} |v-u_0(t)|$ exists, then the number:

$$c_0 = \sup_{t\in[t_0,t_1]} \left|\lambda_0 f_u(x_0,u_0,t) - \psi^{\mathrm{T}}(t) h_u(x_0,u_0,t)\right| \sup_{v\in U}\ \sup_{t\in[t_0,t_1]} |v-u_0(t)|$$

does not depend on s and we have:

$$y^*(s) \geq -\sum_{i=1}^{l}\sum_{k=1}^{p}(s_{ik}^{+}+s_{ik}^{-})\nu(E_i)\ c_0$$

or

$$y^*(s) \geq -c_0 \left\|s\right\|_{L_1}.$$

Switching $s \to (-s)$ we prove that

$$|y^*(s)| \leq c_0 \left\|s\right\|_{L_1},$$

and y^* is continuous on S in L_1-norm. ■

Hence, ρ_0 is the Radon-Nykodim derivative of measure μ with respect to Lebesgue's measure, i.e.:

$$d\mu = \rho_0\ dt$$

and the equation (37) can be written:

$$\begin{aligned}-\psi^{\mathrm{T}}(t) = & \int_t^{t_1}\Big[\lambda_0 f_x(x_0,u_0,\tau) - \psi^{\mathrm{T}}(\tau) h_x(x_0,u_0,\tau) + \rho_0(\tau) g_x(x_0,u_0,\tau)\Big]\ d\tau + \\ & + \int_t^{t_1} q_x(x_0,\tau)\ d\eta + \lambda^{\mathrm{T}}\end{aligned} \tag{39}$$

In the same way, the equation (38) is equivalent to:

$$\int_{t_0}^{t_1}\Big[\lambda_0 f_u(x_0,u_0,t) + g_u(x_0,u_0,t)\rho_0(t) - \psi^{\mathrm{T}}(t) h_u(x_0,u_0,t)\Big]\, v(t)\ dt = f(v),\ \forall v\in L_\infty.$$

Since f is a support functional of integral type of $\mathcal{U}$ at u_0,we obtain (see [13]):

$$\begin{aligned}\Big[\lambda_0 f_u(x_0,u_0,t) + g_u(x_0,u_0,t)\rho_0(t) - \psi^{\mathrm{T}}(t) h_u(x_0,u_0,t)\Big]\,(u-u_0(t)) \geq 0 \quad , \\ \forall u\in U \quad a.e.\ t\in[t_0,t_1]\end{aligned}$$

If we define $\mathcal{H}(x,\psi,u,t) = \psi^{\mathrm{T}} h(x,u,t) + \rho\, g(x,u,t) - \lambda_0 f(x,u,t)$ and substitute in the equations (37) and (38), we obtain the relations (21) and (22). Furthermore, by the complementary slackness relation of the Karush-Kuhn-Tucker theorem, we obtain that $\mu(t)$ and $\eta(t)$ are positive measures with support in the sets:

$$R_0 = \{t \in [t_0, t_1] \mid g(x_0(t), u_0(t), t) = 0_p\},$$

$$R_1 = \{t \in [t_0, t_1] \mid q(x_0(t), t) = 0_q\}$$

and then, ρ_0 must satisfy the conditions:

$$\begin{array}{l} \rho_0(t) \geq 0_p, \quad a.e.\ t \in [t_0, t_1], \\ \rho_0(t)\ g(x_0(t), u_0(t), t) = 0, \quad a.e.\ t \in [t_0, t_1]. \ \blacksquare \end{array}$$

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